Chapter 4: Energy in Quantum Mechanics

Quantum energy teleportation is, of course, about energy.

But in quantum mechanics, energy is not just a number attached to an object. Energy is represented by an operator, determines how states change in time, defines ground and excited states, and can be distributed among parts of a many-body system in subtle ways. In QET, Alice's measurement injects energy into a system, Bob's later operation extracts energy from another region, and the total bookkeeping must remain consistent with ordinary quantum mechanics and energy conservation.

This chapter builds the energy language we need.

We will begin with the Hamiltonian, the quantum observable associated with energy. Then we will study ground states, excited states, time evolution, local energy density, and conservation laws. By the end of the chapter, the central energy question of QET will be clear:

> If the total ground state has the lowest possible energy, how can Bob locally extract energy after Alice performs a distant measurement?

The full answer requires entanglement, locality, and measurement back-action. This chapter prepares the energy side of that answer.

The mathematical framework follows the standard Hilbert-space formulation of quantum mechanics used in undergraduate and graduate texts such as Sakurai and Napolitano (2020), Griffiths and Schroeter (2018), and Nielsen and Chuang (2010).

4.1 Energy as an Observable

In quantum mechanics, an observable is a physical quantity that can be measured. Mathematically, observables are represented by Hermitian operators. An operator hatA is Hermitian if

$$\hat{A}^\dagger = \hat{A},$$

where † means complex conjugate transpose. Hermitian operators have real eigenvalues, which is necessary because measurement results such as energy, position, and spin components are real numbers.

The observable for energy is called the Hamiltonian. We write it as

$$\hat{H}.$$

The Hamiltonian is one of the most important objects in quantum mechanics. It has two central roles:

1. Its measurement outcomes are possible energy values.
2. It generates time evolution.

The second role is especially important. If the state vector of a closed quantum system is |ψ(t)⟩, then its time evolution is governed by the Schrödinger equation,

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle,$$

where hbar is the reduced Planck constant. This equation is one of the basic postulates of nonrelativistic quantum mechanics (Sakurai and Napolitano, 2020).

So, in quantum mechanics, energy is not merely something the system “has.” Energy also determines how the system moves through Hilbert space.

4.2 Energy Eigenstates and Energy Measurements

Suppose the Hamiltonian has eigenvectors |Eₙ⟩ satisfying

$$\hat{H}|E_n\rangle = E_n|E_n\rangle.$$

Here Eₙ is an energy eigenvalue, and |Eₙ⟩ is an energy eigenstate.

If the system is in the state |Eₙ⟩, then an ideal measurement of energy gives the result Eₙ with probability 1. If the system is in a superposition,

$$|\psi\rangle = \sum_n c_n|E_n\rangle,$$

then an energy measurement gives Eₙ with probability

$$p_n = |c_n|^2,$$

assuming the energy eigenstates form an orthonormal basis and there is no degeneracy. This is the Born rule applied to energy measurements.

The expectation value of energy is

$$\langle \hat{H} \rangle_\psi = \langle \psi | \hat{H} | \psi \rangle.$$

For the superposition above,

$$\langle \hat{H} \rangle_\psi = \sum_n |c_n|^2 E_n.$$

This expectation value is not necessarily one of the possible measurement results. It is the average result obtained from many identically prepared systems.

For a mixed state described by a density matrix ρ, the expectation value is

$$\langle \hat{H} \rangle_\rho = \mathrm{Tr}(\rho \hat{H}).$$

This formula will be used throughout QET because measurements often produce mixed states when we ignore, average over, or condition on measurement outcomes.

4.3 A Simple Example: A Qubit Hamiltonian

Consider a single qubit with basis states |0⟩ and |1⟩. Let its Hamiltonian be

$$\hat{H} = \hbar\omega |1\rangle\langle 1|.$$

This means

$$\hat{H}|0\rangle = 0,$$

and

$$\hat{H}|1\rangle = \hbar\omega|1\rangle.$$

So |0⟩ has energy 0, and |1⟩ has energy hbarω.

A general pure qubit state can be written as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,$$

where

$$|\alpha|^2 + |\beta|^2 = 1.$$

The expected energy is

$$\langle \hat{H} \rangle_\psi = \hbar\omega|\beta|^2.$$

This example shows an important idea: the expected energy depends on how much of the state lies in the excited-energy component.

If $|\beta|^2 = 0$, the expected energy is 0. If $|\beta|^2 = 1$, the expected energy is hbarω. If $|\beta|^2 = 1/2$, the expected energy is hbarω/2.

This simple model will help us later when we study how measurements can inject energy.

4.4 Ground States and Excited States

The ground-state energy is the lowest possible energy of a quantum system. If the Hamiltonian has a lowest eigenvalue E_0, then

$$\hat{H}|g\rangle = E_0|g\rangle$$

means |g⟩ is a ground state.

Any state with energy greater than E_0 is called an excited state. If

$$\hat{H}|E_n\rangle = E_n|E_n\rangle,$$

and

$$E_n > E_0,$$

then |Eₙ⟩ is an excited energy eigenstate.

A system may have one ground state or several. If there is exactly one linearly independent ground state, the ground state is called nondegenerate. If there are multiple independent ground states with the same lowest energy, the ground state is degenerate.

For many QET discussions, we imagine a many-body system prepared in a ground state. This matters because a ground state cannot have its total energy lowered by any physical operation that simply changes the state while leaving the Hamiltonian fixed. The ground state already has the minimum possible energy.

More precisely, suppose hatH has lowest eigenvalue E₀, and |g⟩ is a ground state. For any normalized state |φ⟩,

$$\langle \phi | \hat{H} | \phi \rangle \geq E_0.$$

If we apply a unitary operation hatU to the ground state, the new state is

$$|\phi\rangle = \hat{U}|g\rangle.$$

Therefore,

$$\langle g | \hat{U}^\dagger \hat{H} \hat{U} | g \rangle \geq E_0.$$

So a unitary operation cannot turn the ground state into a state with lower total energy.

This simple fact will become crucial in QET. Bob cannot extract energy from the untouched ground state by acting locally. Something must happen first. In Hotta’s QET protocol, Alice’s local measurement changes the state and injects energy; Bob then uses Alice’s classical information to perform a conditional local operation that lowers the system’s energy near him (Hotta, 2008).

4.5 Shifting the Zero of Energy

Energy differences are usually more physically important than the absolute zero of energy. If we add a constant C to the Hamiltonian,

$$\hat{H}' = \hat{H} + C\hat{I},$$

then every energy eigenvalue shifts by C:

$$E'_n = E_n + C.$$

But energy differences remain the same:

$$E'_n - E'_m = E_n - E_m.$$

The time evolution changes only by an overall phase:

$$e^{-i\hat{H}'t/\hbar} = e^{-iCt/\hbar}e^{-i\hat{H}t/\hbar}.$$

For a pure state, an overall phase has no observable effect. Therefore, shifting all energies by a constant does not change ordinary measurement probabilities or dynamics.

In QET, it is common to choose the zero of energy so that the ground-state energy is

$$E_0 = 0.$$

Then the Hamiltonian is positive semidefinite:

$$\langle\psi|\hat{H}|\psi\rangle \geq 0$$

for every normalized state $|\psi\rangle$.

This convention makes energy bookkeeping clearer. If the system begins in the ground state, its total energy is 0. If Alice's measurement injects energy, the total energy becomes positive. If Bob extracts energy, the system's total energy decreases, but it cannot go below 0.

4.6 The Energy Gap

The energy gap is the energy difference between the ground state and the first excited level. If the ground-state energy is E_0 and the first excited energy is E_1, then the gap is

$$\Delta = E_1 - E_0.$$

A system is called gapped if $\Delta > 0$ in the relevant finite system or in the thermodynamic limit being considered. It is called gapless if arbitrarily low-energy excitations exist above the ground state in the large-system limit.

For a single qubit with Hamiltonian

$$\hat{H} = \hbar\omega|1\rangle\langle 1|,$$

the ground energy is 0, the excited energy is hbarω, and the gap is

$$\Delta = \hbar\omega.$$

In many-body systems, the energy gap strongly affects correlations and dynamics. For example, in many lattice systems with local interactions, gapped ground states often have correlations that decay with distance, while critical gapless systems can have longer-ranged correlations. The detailed relationship depends on the system, but the distinction between gapped and gapless behavior is central in quantum many-body physics (Sakurai and Napolitano, 2020).

For QET, this matters because Bob’s extractable energy depends on correlations between Alice’s region and Bob’s region. If correlations decay quickly with distance, QET energy gain typically becomes small at large separation. If correlations are long-ranged, the distance dependence can be different. We will return to this in Chapter 10 when we study spin chains.

4.7 Time Evolution from the Hamiltonian

If the Hamiltonian is time-independent, the Schrödinger equation has the formal solution

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle.$$

The operator

$$\hat{U}(t) = e^{-i\hat{H}t/\hbar}$$

is unitary:

$$\hat{U}^\dagger(t)\hat{U}(t) = \hat{I}.$$

Unitary time evolution preserves normalization and inner products. Therefore, probabilities remain consistent over time.

For a density matrix, the time evolution is

$$\rho(t) = \hat{U}(t)\rho(0)\hat{U}^\dagger(t).$$

Differentiating gives the von Neumann equation,

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[\hat{H}, \rho],$$

where

$$[\hat{H}, \rho] = \hat{H}\rho - \rho\hat{H}$$

is the commutator.

The commutator measures noncommutativity. If two operators commute, their order does not matter. If they do not commute, their order matters. In quantum mechanics, noncommutativity is often the source of change and uncertainty.

4.8 Conservation of Total Energy

For a closed system with a time-independent Hamiltonian, the expected total energy is conserved.

Let

$$E(t) = \mathrm{Tr}(\rho(t)\hat{H}).$$

Using the von Neumann equation,

$$\frac{dE}{dt} = \mathrm{Tr}\left(\frac{d\rho}{dt}\hat{H}\right) = -\frac{i}{\hbar}\,\mathrm{Tr}([\hat{H},\rho]\hat{H}).$$

Expanding the commutator,

$$\frac{dE}{dt} = -\frac{i}{\hbar}\,\mathrm{Tr}(\hat{H}\rho\hat{H} - \rho\hat{H}^2).$$

Using the cyclic property of the trace,

$$\mathrm{Tr}(\hat{H}\rho\hat{H}) = \mathrm{Tr}(\rho\hat{H}^2).$$

Therefore,

$$\frac{dE}{dt} = 0.$$

So, for a closed system with no explicit time dependence in hatH, total energy is conserved.

This does not mean that energy cannot move around inside the system. It means the total amount stays constant. Local energy can increase in one region and decrease in another, as long as the total is unchanged.

That distinction is essential for QET. Alice can inject energy locally. Later, energy can propagate through the system in the ordinary dynamical way. But QET is different from simply waiting for Alice’s injected energy to travel to Bob. In QET, Bob may extract energy before any physical energy carrier from Alice’s region could have reached him, provided he has received Alice’s classical message through an allowed channel. The protocol remains consistent with causality because Bob needs the classical information, and that information cannot travel faster than light.

4.9 When Energy Is Not Conserved: External Driving

Energy conservation for the system alone can fail if the Hamiltonian depends explicitly on time:

$$\hat{H} = \hat{H}(t).$$

A time-dependent Hamiltonian usually means the system is being driven by something external. For example, a laser pulse may drive an atom, or a control field may rotate a qubit.

The expected energy is

$$E(t) = \mathrm{Tr}(\rho(t)\hat{H}(t)).$$

Then

$$\frac{dE}{dt} = \mathrm{Tr}\left(\frac{d\rho}{dt}\hat{H}\right) + \mathrm{Tr}\left(\rho\frac{\partial \hat{H}}{\partial t}\right).$$

The first term vanishes for unitary evolution generated by hatH(t), by the same trace-cyclic argument as before. Thus,

$$\frac{dE}{dt} = \mathrm{Tr}\left(\rho\frac{\partial \hat{H}}{\partial t}\right).$$

This term represents energy exchanged with the external controller.

In QET, Alice’s measurement device and Bob’s operation device are not free. They interact with the quantum system. Alice’s apparatus supplies energy during measurement. Bob’s apparatus can receive energy when Bob extracts it. A correct thermodynamic description must include these devices, not just the many-body system.

This is why QET does not violate energy conservation. Energy is not created from information alone. Instead, information allows Bob to choose an operation that transfers energy from the many-body system into his local device.

4.10 Measurement Can Change Energy

In Chapter 2, we saw that a quantum measurement can change the state. Since energy depends on the state, a measurement can change the expected energy.

Let the system initially have density matrix ρ. Suppose a projective measurement has projectors \hatPμ\, where μ labels the measurement outcome. The probability of outcome μ is

$$p_\mu = \mathrm{Tr}(\hat{P}_\mu \rho).$$

If outcome μ is obtained, the post-measurement state is

$$\rho_\mu = \frac{\hat{P}_\mu \rho \hat{P}_\mu}{p_\mu}.$$

If we ignore the outcome, the average post-measurement state is

$$\rho' = \sum_\mu p_\mu \rho_\mu = \sum_\mu \hat{P}_\mu \rho \hat{P}_\mu.$$

The average energy change caused by the measurement is

$$\Delta E_{\mathrm{meas}} = \mathrm{Tr}(\rho' \hat{H}) - \mathrm{Tr}(\rho \hat{H}).$$

If every measurement projector commutes with the Hamiltonian,

$$[\hat{P}_\mu, \hat{H}] = 0$$

for all μ, then the measurement does not change the average energy. Such a measurement is compatible with energy.

But if the measurement projectors do not commute with the Hamiltonian, the measurement can change the energy.

This is not mysterious. A real measurement requires an interaction between the system and a measuring apparatus. If the measurement disturbs variables that matter for energy, then energy can be exchanged with the apparatus.

4.11 Example: Measuring a Qubit Injects Energy

Return to the qubit Hamiltonian

$$\hat{H} = \hbar\omega|1\rangle\langle 1|.$$

The ground state is $|0\rangle$, with energy 0.

Now measure the qubit in the hatσ_x basis. The two measurement states are

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}},$$

and

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

The projectors are

$$\hat{P}_+ = |+\rangle\langle +|,$$

and

$$\hat{P}_- = |-\rangle\langle -|.$$

If the initial state is $|0\rangle$, then

$$p_+ = |\langle +|0\rangle|^2 = \frac{1}{2},$$

and

$$p_- = |\langle -|0\rangle|^2 = \frac{1}{2}.$$

After the measurement, the state is either |+⟩ or |-⟩. Each has expected energy

$$\langle +|\hat{H}|+\rangle = \frac{\hbar\omega}{2},$$

and

$$\langle -|\hat{H}|-\rangle = \frac{\hbar\omega}{2}.$$

Therefore the average post-measurement energy is

$$E' = \frac{1}{2}\frac{\hbar\omega}{2} + \frac{1}{2}\frac{\hbar\omega}{2} = \frac{\hbar\omega}{2}.$$

The initial energy was 0. Thus,

$$\Delta E_{\text{meas}} = \frac{\hbar\omega}{2}.$$

The measurement injected energy into the qubit.

This example is small, but it contains the core mechanism used by Alice in QET. Alice performs a local measurement that does not commute with the full Hamiltonian. The measurement disturbs the ground state and injects positive energy.

4.12 Many-Body Hamiltonians

QET requires many-body systems. A many-body system is a system with many interacting parts: spins on a lattice, atoms in an optical trap, coupled oscillators, or field modes.

A typical lattice Hamiltonian has the form

$$\hat{H} = \sum_n \hat{h}_n + \sum_{\langle n,m\rangle} \hat{v}_{nm}.$$

Here:

- hath$_n$ is an on-site energy term for site n.

- hatv$_{nm}$ is an interaction energy term between sites n and m.

- ⟨ n,m⟩ means the sum is over neighboring pairs or some chosen interaction graph.

For example, a simple spin-chain Hamiltonian might be

$$\hat{H} = -J \sum_n \hat{\sigma}_n^z \hat{\sigma}_{n+1}^z - h \sum_n \hat{\sigma}_n^x.$$

Here hatσ_n^z and hatσ_n^x are Pauli operators acting on site n. The first term couples neighboring spins. The second term represents a transverse field.

Spin-chain models of this general kind are widely used in quantum many-body physics and also appear in QET studies, including Hotta's early spin-chain protocol (Hotta, 2008).

4.13 Local Energy

In a many-body system, we often want to discuss energy in a region. This leads to the idea of local energy.

Suppose a chain has Hamiltonian

$$\hat{H} = \sum_n \hat{T}_n,$$

where hatT$_n$ is called a local energy density operator near site n. Then the energy in a region R may be written as

$$\hat{H}_R = \sum_{n \in R} \hat{T}_n.$$

The expected energy in region R is

$$E_R = \langle \hat{H}_R \rangle = \mathrm{Tr}(\rho \hat{H}_R).$$

This looks straightforward, but there is a subtlety: local energy decompositions are not always unique.

For example, suppose two neighboring spins interact through

$$\hat{v}_{12} = J\hat{\sigma}_1^z\hat{\sigma}_2^z.$$

Should this interaction energy belong to site 1, to site 2, or half to each? One common convention is to split it evenly:

$$\hat{T}_1 = \hat{h}_1 + \frac{1}{2}\hat{v}_{12},$$

$$\hat{T}_2 = \hat{h}_2 + \frac{1}{2}\hat{v}_{12}.$$

But another convention may be more useful in another context. The total Hamiltonian is physical, while the precise division into local pieces may depend on modeling choices.

This does not make local energy meaningless. It means we must state our convention clearly.

In QET papers, one often chooses local energy density operators so that the ground state has zero expectation value locally:

$$\langle g|\hat{T}_n|g\rangle = 0$$

for each n, while also choosing

$$\hat{H} = \sum_n \hat{T}_n$$

and

$$\langle g|\hat{H}|g\rangle = 0.$$

This convention makes it easy to identify where positive and negative local energy densities appear during the protocol. Hotta's original spin-chain formulation uses this kind of local energy bookkeeping (Hotta, 2008).

4.14 Local Energy Can Be Negative

If the total Hamiltonian has been shifted so that

$$\hat{H} \geq 0,$$

then the total expected energy is never negative:

$$\langle \hat{H} \rangle \geq 0.$$

However, a local energy density expectation value can be negative:

$$\langle \hat{T}_n \rangle < 0.$$

This may seem strange at first. How can a part have negative energy if the whole system has nonnegative energy?

The answer is that local energy operators hatTₙ are not necessarily positive operators. The sum

$$\hat{H} = \sum_n \hat{T}_n$$

may be positive even though individual terms can have negative expectation values.

A simple analogy is ordinary arithmetic:

$$5 = 8 + (-3).$$

The total is positive, but one contribution is negative.

In quantum systems, this is more than arithmetic. Interaction terms can contribute negative energy depending on correlations between neighboring parts. In field theory, local energy density can also have negative expectation values in some quantum states, although such negative energies are constrained by deeper principles. We will discuss this more carefully in Chapter 13.

For QET, negative local energy density is important because Bob’s extraction operation often leaves behind a region whose local energy expectation is below its ground-state reference value. The total energy remains nonnegative because Alice’s earlier measurement injected enough positive energy elsewhere.

4.15 Energy Flow and Local Conservation

Total energy conservation says

$$\frac{d}{dt}\langle \hat{H} \rangle = 0$$

for a closed system with time-independent hatH. But local energy may change.

Let hatT_n be the local energy density near site n. In the Heisenberg picture, operators evolve in time while states remain fixed. For an operator with no explicit time dependence,

$$\frac{d\hat{T}_n}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{T}_n].$$

Taking expectation values,

$$\frac{d}{dt}\langle \hat{T}_n \rangle = \frac{i}{\hbar}\langle [\hat{H}, \hat{T}_n] \rangle.$$

For many nearest-neighbor lattice systems, this equation can be written in a form resembling a continuity equation:

$$\frac{d}{dt}\langle \hat{T}_n \rangle = \langle \hat{J}_{n-1} \rangle - \langle \hat{J}_n \rangle.$$

Here hatJₙ is an energy-current operator representing energy flow from site n to site n+1.

This equation says:

> The energy near site n changes because energy flows in from one side and out through the other.

This is the quantum lattice version of a local conservation law. The exact definition of hatJₙ depends on the Hamiltonian and the chosen local energy density.

This idea will matter later when we distinguish QET from ordinary energy transport. If Alice injects energy into her region, that energy can later flow through the system according to the Hamiltonian dynamics. But QET is not merely the arrival of that energy current at Bob. QET uses pre-existing ground-state correlations plus Alice's classical information.

4.16 Local Operations and Energy Change

A local operation is an operation applied only to a limited region of a composite system.

Suppose Bob acts on region B with a unitary operator hatU₍B₎. Mathematically, on the full Hilbert space this means

$$\hat{U} = \hat{I}_{\bar{B}} \otimes \hat{U}_B,$$

where barB denotes everything outside Bob's region.

If the state before Bob's operation is ρ, then after the operation,

$$\rho' = \hat{U}\rho\hat{U}^\dagger.$$

The total energy change of the many-body system is

$$\Delta E_B = \mathrm{Tr}(\rho'\hat{H}) - \mathrm{Tr}(\rho\hat{H}).$$

Using cyclicity of the trace,

$$\Delta E_B = \mathrm{Tr}\left(\rho(\hat{U}^\dagger \hat{H}\hat{U} - \hat{H})\right).$$

If

$$\Delta E_B < 0,$$

then the system has lost energy. If Bob's apparatus gains that energy, Bob has extracted

$$E_{\mathrm{out}} = -\Delta E_B > 0.$$

This is the mathematical meaning of energy extraction in QET.

Notice an important point: Bob's operation changes only terms in the Hamiltonian that fail to commute with hatU. Since hatU acts locally, only Hamiltonian terms supported in or near Bob's region can contribute. Distant Hamiltonian terms commute with Bob's operation and do not directly change.

This is how QET remains local. Bob does not reach across space and pull energy from Alice's site. He acts locally. The possibility of energy extraction comes from how the many-body state, correlations, and local Hamiltonian terms combine.

4.17 Why Bob Cannot Extract Energy from the Unmeasured Ground State

Now let us use the ground-state property in a QET-like setting.

Assume the many-body system begins in a ground state |g⟩, and choose the zero of energy so that

$$\hat{H}|g\rangle = 0.$$

Then for any normalized state |φ⟩,

$$\langle\phi|\hat{H}|\phi\rangle \geq 0.$$

Suppose Bob applies any local unitary hatU(B). The final state is

$$|\phi\rangle = \hat{U}_B|g\rangle.$$

Therefore,

$$\langle g|\hat{U}_B^\dagger \hat{H}\hat{U}_B|g\rangle \geq 0.$$

The initial energy was

$$\langle g|\hat{H}|g\rangle = 0.$$

So Bob's energy change is

$$\Delta E_B = \langle g|\hat{U}_B^\dagger \hat{H}\hat{U}_B|g\rangle - 0 \geq 0.$$

Bob cannot lower the system's energy below the ground state. Thus, he cannot extract positive energy from the untouched ground state by local unitary action alone.

This is one of the clearest ways to see why Alice's measurement is necessary.

Alice's measurement changes the state from the ground state to a higher-energy state. It also produces a classical outcome correlated with fluctuations in Bob's region. Bob's conditional operation can then lower the system's energy locally. Without Alice's outcome, Bob would not know which operation to perform.

This is the core energy logic behind Hotta's QET protocol (Hotta, 2008).

4.18 Measurement Energy Injection in a Many-Body Ground State

Let the initial many-body state be the ground state density matrix

$$\rho_g = |g\rangle\langle g|.$$

Alice performs a local measurement in region A. Let the measurement operators be \hatMμ\, where μ labels Alice's outcome. They satisfy

$$\sum_{\mu} \hat{M}_{\mu}^{\dagger} \hat{M}_{\mu} = \hat{I}.$$

The probability of outcome μ is

$$p_{\mu} = \mathrm{Tr}(\hat{M}_{\mu} \rho_g \hat{M}_{\mu}^{\dagger}).$$

The conditional post-measurement state is

$$\rho_{\mu} = \frac{\hat{M}_{\mu} \rho_g \hat{M}_{\mu}^{\dagger}}{p_{\mu}}.$$

If we average over Alice's outcomes, the post-measurement state is

$$\rho_A = \sum_{\mu} \hat{M}_{\mu} \rho_g \hat{M}_{\mu}^{\dagger}.$$

The energy injected by Alice, on average, is

$$E_A = \mathrm{Tr}(\rho_A \hat{H}) - \mathrm{Tr}(\rho_g \hat{H}).$$

If we set the ground energy to zero, then

$$\mathrm{Tr}(\rho_g \hat{H}) = 0,$$

so

$$E_A = \mathrm{Tr}(\rho_A \hat{H}).$$

Because hatH≥ 0,

$$E_A \geq 0.$$

Thus Alice’s measurement cannot reduce the total energy below the ground state. On average, it injects nonnegative energy.

In a nontrivial QET protocol, this injected energy is not just a nuisance. It is part of the physical bookkeeping that makes Bob’s later extraction possible.

4.19 Conditional Energy Extraction

After Alice measures, she sends the classical outcome μ to Bob. Bob then applies an operation depending on μ. In the simplest case, Bob applies a unitary

$$\hat{U}_\mu.$$

The final averaged state is

$$\rho_{\text{final}} = \sum_\mu p_\mu \hat{U}_\mu \rho_\mu \hat{U}_\mu^\dagger.$$

Equivalently,

\[ρ□final□ = ∑_μ hatU_μ hatM_μ ρ□g□ hatM_μ^□ \

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