Chapter 2: Quantum States, Observables, and Measurements

Quantum energy teleportation begins with a measurement.

That may sound ordinary. In a laboratory, measurement is how we learn what a system is doing. But in quantum mechanics, measurement is not merely passive observation. A measurement can change the state of the system being measured. In QET, this fact is not a technical detail—it is the starting point. Alice's local measurement changes the many-body quantum system, injects energy, and produces a classical outcome. Bob later uses that outcome to choose a local operation that can extract energy from his region.

Before we can understand that process, we need the basic mathematical language of quantum states, observables, and measurements. This chapter builds that language carefully.

We will mostly use finite-dimensional examples, especially qubits and spins. A qubit is a quantum system whose state space is two-dimensional. Spin-chain QET models often use qubits or spin-frac12 particles, so this is not just a toy setting. The same concepts extend to infinite-dimensional systems, such as harmonic oscillators and quantum fields, but those require additional mathematical care.

The standard modern formulation used here follows the usual Hilbert-space description of quantum mechanics and quantum information theory (Sakurai and Napolitano, 2020; Nielsen and Chuang, 2010).

2.1 The State of a Quantum System

In classical physics, the state of a simple particle might be described by its position and momentum. If we know both exactly, then, at least in Newtonian mechanics, we can predict its future motion.

Quantum mechanics uses a different kind of state.

A pure quantum state is represented by a vector in a mathematical space called a Hilbert space. For our purposes, a Hilbert space is a complex vector space with an inner product. Let us unpack that.

A vector space is a set of objects that can be added together and multiplied by numbers. A complex vector space allows multiplication by complex numbers such as

$$a + ib,$$

where $i^2=-1$. An inner product is a rule that takes two vectors and produces a complex number. It allows us to define lengths, angles, orthogonality, and probabilities.

In quantum notation, called Dirac notation, a state vector is written as a ket:

$$|\psi\rangle.$$

The corresponding dual vector is written as a bra:

$$\langle\psi|.$$

The inner product between two states $|\varphi\rangle$ and $|\psi\rangle$ is

$$\langle\phi|\psi\rangle.$$

The length squared of $|\psi\rangle$ is

$$\langle\psi|\psi\rangle.$$

A physical pure state must be normalized:

$$\langle\psi|\psi\rangle = 1.$$

This normalization condition is necessary because probabilities must add to 1.

2.2 A First Example: The Qubit

The simplest important quantum system is a qubit. Its Hilbert space is two-dimensional. We usually choose two basis states,

$$|0\rangle \quad \text{and} \quad |1\rangle.$$

A basis is a set of vectors from which every vector in the space can be built. For a qubit, any pure state can be written as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,$$

where α and β are complex numbers.

The normalization condition is

$$|\alpha|^2 + |\beta|^2 = 1.$$

The numbers α and β are called probability amplitudes. They are not themselves probabilities. Their squared magnitudes become probabilities when we measure in the \|0⟩, |1⟩\ basis.

For example,

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

is normalized because

$$\left|\frac{1}{\sqrt{2}}\right|^2 + \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2} + \frac{1}{2} = 1.$$

If we measure this state in the |0⟩, |1⟩ basis, the probability of outcome 0 is frac12, and the probability of outcome 1 is frac12. This probability rule is the Born rule, one of the central postulates of quantum mechanics (Sakurai and Napolitano, 2020).

2.3 Global Phase and Physical States

There is a subtle but important point: not every difference between two state vectors represents a physical difference.

The states

$$|\psi\rangle$$

and

$$e^{i\theta}|\psi\rangle$$

represent the same physical pure state, where e^(iθ) is a complex number with magnitude 1. This factor is called a global phase.

For example,

$$|0\rangle \quad \text{and} \quad -|0\rangle$$

represent the same physical state. No measurement can distinguish them.

However, relative phase does matter. Compare

$$|\psi_+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

with

$$|\psi_-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

These are physically different states. In fact, they are orthogonal:

$$\langle\psi_+|\psi_-\rangle = 0.$$

They can be perfectly distinguished by a measurement in the basis

$$\left\{ \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right\}.$$

This matters for QET because useful quantum correlations are often encoded not only in probabilities, but also in phases and coherences.

2.4 Observables: Quantities We Can Measure

An observable is a physical quantity that can be measured, such as position, momentum, spin, or energy.

In the Hilbert-space formulation of quantum mechanics, observables are represented by Hermitian operators. An operator is a mathematical object that acts on a state vector and produces another vector. If A is an operator, then

$$A|\psi\rangle$$

is another vector in the Hilbert space.

An operator A is Hermitian if

$$A^\dagger = A,$$

where A^† is the adjoint of A. Hermitian operators have real eigenvalues, which is why they are suitable for representing measurable quantities (Sakurai and Napolitano, 2020).

An eigenvector of A is a vector |a⟩ satisfying

$$A|a\rangle = a|a\rangle.$$

The number a is the corresponding eigenvalue. When we measure an observable, the possible outcomes are the eigenvalues of its operator, at least for ideal projective measurements.

2.5 Example: Measuring Spin with Pauli Operators

For a qubit, three important observables are the Pauli operators:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In the basis \|0⟩, |1⟩\, the operator Z satisfies

$$Z|0\rangle = |0\rangle,$$

and

$$Z|1\rangle = -|1\rangle.$$

So |0⟩ is an eigenstate of Z with eigenvalue +1, and |1⟩ is an eigenstate of Z with eigenvalue -1.

If a qubit is in the state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,$$

then a measurement of Z gives outcome +1 with probability

$$|\alpha|^2,$$

and outcome -1 with probability

$$|\beta|^2.$$

This example already shows a key quantum idea: before measurement, the state may not have a definite value of the observable being measured.

2.6 Expectation Values

Often we do not only care about individual measurement outcomes. We care about the average result over many repetitions.

The expectation value of an observable A in a pure state |ψ⟩ is

$$\langle A \rangle_\psi = \langle \psi | A | \psi \rangle.$$

This is the average value we would obtain if we prepared the same state many times and measured A each time.

For example, let

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle.$$

The expectation value of Z is

$$\langle Z \rangle_\psi = \langle \psi | Z | \psi \rangle.$$

Using

$$Z|0\rangle = |0\rangle, \quad Z|1\rangle = -|1\rangle,$$

we get

$$\langle Z \rangle_\psi = |\alpha|^2 - |\beta|^2.$$

If

$$|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}},$$

then

$$\langle Z \rangle_\psi = \frac{1}{2} - \frac{1}{2} = 0.$$

This does not mean that a single measurement gives 0. A measurement of Z gives either +1 or -1. The expectation value 0 means that the average of many such measurements is 0.

This distinction is especially important when we later discuss energy. If H is the Hamiltonian, the energy expectation value in state $|\psi\rangle$ is

$$\langle H\rangle_\psi = \langle\psi|H|\psi\rangle.$$

In QET, we track changes in energy expectation values before and after local measurements and local operations.

2.7 Projective Measurements

The simplest idealized kind of quantum measurement is a projective measurement.

Suppose an observable A has spectral decomposition

$$A = \sum_a a P_a,$$

where a labels possible measurement outcomes and P_a is the projector onto the subspace associated with outcome a. A projector is an operator satisfying

$$P_a^2 = P_a, \quad P_a^\dagger = P_a.$$

The projectors of a projective measurement satisfy

$$P_a P_b = 0 \quad \text{for } a \neq b,$$

and

$$\sum_a P_a = I,$$

where I is the identity operator.

If the system is initially in pure state |ψ⟩, then the probability of outcome a is

$$p(a) = \langle\psi|P_a|\psi\rangle.$$

If outcome a occurs, the post-measurement state is

$$|\psi_a\rangle = \frac{P_a|\psi\rangle}{\sqrt{p(a)}}.$$

This rule is often called the projection postulate in the case of ideal projective measurements, and it is a standard part of the textbook formulation of quantum measurement (von Neumann, 1955; Sakurai and Napolitano, 2020).

2.8 Example: Projective Measurement of a Qubit

Consider

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle.$$

We measure Z. The projectors are

$$P_0 = |0\rangle\langle 0|,$$

$$P_1 = |1\rangle\langle 1|.$$

The probability of outcome 0 is

$$p(0) = \langle\psi|P_0|\psi\rangle = \frac{3}{4}.$$

The probability of outcome 1 is

$$p(1) = \langle\psi|P_1|\psi\rangle = \frac{1}{4}.$$

If outcome 0 occurs, the state becomes

$$|\psi_0\rangle = |0\rangle.$$

If outcome 1 occurs, the state becomes

$$|\psi_1\rangle = |1\rangle.$$

So the measurement does two things:

1. It produces a classical outcome.
2. It changes the quantum state.

The second part is crucial for QET. Alice's measurement is not merely her learning something. It is a physical operation on the many-body system.

2.9 Pure States Are Not Enough: Density Matrices

Pure states are important, but they do not describe every situation.

Sometimes a system is in a state of classical uncertainty. For example, suppose a qubit is prepared as follows:

- with probability frac12, it is prepared in $|0\rangle$;
- with probability frac12, it is prepared in $|1\rangle$.

This is not the same as the pure state

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}}.$$

The pure state has quantum coherence between $|0\rangle$ and $|1\rangle$. The random preparation does not.

To describe both pure states and probabilistic mixtures, we use a density matrix, also called a density operator.

If a system is in pure state $|\psi_j\rangle$ with probability p_j, then its density matrix is

$$\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|.$$

The probabilities satisfy

$$p_j \geq 0, \quad \sum_j p_j = 1.$$

A valid density matrix satisfies three conditions:

$$\rho^\dagger = \rho,$$

$$\rho \geq 0,$$

and

$$\mathrm{Tr}(\rho) = 1.$$

Here $\rho \geq 0$ means that ρ is positive semidefinite: for every vector $|\varphi\rangle$,

$$\langle \phi | \rho | \phi \rangle \geq 0.$$

The trace Tr is the sum of diagonal entries in any orthonormal basis. Density matrices are central in quantum information theory because they describe pure states, mixed states, subsystems, measurement outcomes, and open-system dynamics in one unified language (Nielsen and Chuang, 2010).

2.10 Pure Density Matrices and Mixed Density Matrices

Every pure state $|\psi\rangle$ can be written as a density matrix:

$$\rho = |\psi\rangle\langle\psi|.$$

For example,

$$|0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

A mixed state cannot be written as one projector onto a single state vector. For example, the fifty-fifty mixture of $|0\rangle$ and $|1\rangle$ is

$$\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|.$$

In matrix form,

$$\rho = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} = \frac{I}{2}.$$

This is called the maximally mixed state of a qubit. It represents complete uncertainty about the outcome of any spin measurement direction.

Now compare it with the pure state

$$|\psi_+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}.$$

Its density matrix is

$$|\psi_+\rangle\langle\psi_+| = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

The diagonal entries are frac12 and frac12, just like the maximally mixed state. But the off-diagonal entries are nonzero. These off-diagonal entries represent coherence between the basis states.

Coherence is one reason quantum systems behave differently from classical random systems.

2.11 Expectation Values with Density Matrices

For a density matrix ρ, the expectation value of an observable A is

$$\langle A\rangle_\rho = \mathrm{Tr}(\rho A).$$

This formula includes pure states as a special case. If

$$\rho = |\psi\rangle\langle\psi|,$$

then

$$\mathrm{Tr}(\rho A) = \mathrm{Tr}(|\psi\rangle\langle\psi|A) = \langle\psi|A|\psi\rangle.$$

For energy, if H is the Hamiltonian, then

$$E = \mathrm{Tr}(\rho H).$$

Later, a QET protocol will often be described by comparing the energy expectation before and after a measurement:

$$\Delta E = \mathrm{Tr}(\rho_{\mathrm{after}} H) - \mathrm{Tr}(\rho_{\mathrm{before}} H).$$

This is not a metaphor. It is the mathematical quantity used to track energy injection and energy extraction.

2.12 Measurements Written with Density Matrices

Let us rewrite projective measurement using density matrices.

Suppose we perform a projective measurement with projectors P_a\ on a state ρ. The probability of outcome a is

$$p(a) = \mathrm{Tr}(P_a \rho).$$

If outcome a occurs, the post-measurement state is

$$\rho_a = \frac{P_a \rho P_a}{p(a)}.$$

If we perform the measurement but ignore the outcome, the final state is

$$\rho' = \sum_a P_a \rho P_a.$$

This distinction between selective and nonselective measurement is essential.

A selective measurement means that we condition on a particular outcome. For example, “the state after Alice obtained outcome a=+1.”

A nonselective measurement means that the measurement happened, but we do not condition on which outcome occurred. For example, “the state after Alice measured, but before we read or use her result.”

QET depends on selective information. Bob’s operation is chosen according to Alice’s actual outcome. If Bob does not receive the outcome, he cannot perform the correct feedback operation.

2.13 Measurements More General Than Projective Measurements

Projective measurements are important, but real measurements are often more general. They may be noisy, weak, indirect, or implemented by coupling the system to an apparatus.

The general language for quantum measurement uses POVMs and Kraus operators.

A POVM is a positive operator-valued measure. In finite dimensions, it is a collection of positive operators

$$\{E_m\}$$

satisfying

$$E_m \geq 0,$$

and

$$\sum_m E_m = I.$$

The label m denotes the measurement outcome.

If the system is in state ρ, the probability of outcome m is

$$p(m) = \mathrm{Tr}(E_m \rho).$$

The operators E_m are called effects. They determine the probabilities of the outcomes.

But a POVM alone does not fully determine the post-measurement state. To describe how the state changes, we use measurement operators or Kraus operators M_m\, satisfying

$$E_m = M_m^\dagger M_m,$$

and

$$\sum_m M_m^\dagger M_m = I.$$

If outcome m occurs, the post-measurement state is

$$\rho_m = \frac{M_m \rho M_m^\dagger}{p(m)},$$

where

$$p(m) = \mathrm{Tr}(M_m^\dagger M_m \rho).$$

If the outcome is ignored, the post-measurement state is

$$\rho' = \sum_m M_m \rho M_m^\dagger.$$

This operator-sum description of quantum operations is standard in quantum information theory and originates from the general theory of quantum operations developed by Kraus and others (Kraus, 1983; Nielsen and Chuang, 2010).

2.14 Example: An Unsharp Qubit Measurement

A projective measurement of Z forces the state into either $|0\rangle$ or $|1\rangle$. But a real measurement may be weaker. It may give partial information without fully collapsing the state.

One simple POVM for an unsharp Z-measurement is

$$E_+ = \frac{1}{2}(I + \eta Z),$$

$$E_- = \frac{1}{2}(I - \eta Z),$$

where

$$0 \leq \eta \leq 1.$$

The parameter η measures the sharpness of the measurement.

If $\eta=1$, then

$$E_+ = |0\rangle\langle 0|, \quad E_- = |1\rangle\langle 1|,$$

so the measurement is sharp.

If $\eta=0$, then

$$E_+ = E_- = \frac{I}{2}.$$

Then the outcome is just a fair random label and tells us nothing about the state.

For

$$0 < \eta < 1,$$

the measurement gives partial information. For a state ρ, the probabilities are

$$p(+) = \mathrm{Tr}(E_+\rho),$$

$$p(-) = \mathrm{Tr}(E_-\rho).$$

This kind of generalized measurement is useful because QET protocols do not require us to imagine only perfectly ideal measurements. Later, when we discuss noise and experimental platforms, POVMs will help describe imperfect measurement devices.

2.15 Measurement as Physical Disturbance

It is tempting to think of measurement as “finding out what was already there.” Sometimes that intuition is useful, but in quantum mechanics it is incomplete.

If a system is already in an eigenstate of the measured observable, then an ideal projective measurement of that observable does not change the state. For example, if the qubit is in $|0\rangle$, measuring Z gives outcome +1 with probability 1, and the state remains $|0\rangle$.

But if the system is not in an eigenstate, measurement generally changes it.

For example, start with

$$|\psi_+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}.$$

Measure Z. The outcomes +1 and -1 occur with probabilities frac12 and frac12. After the measurement, the state is either $|0\rangle$ or $|1\rangle$.

If we ignore the outcome, the density matrix becomes

$$\rho' = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{I}{2}.$$

Before measurement, the density matrix was

$$\rho = |\psi_+\rangle\langle\psi_+| = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

After nonselective measurement, it is

$$\rho' = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The measurement has removed the off-diagonal coherence in the Z-basis. This process is called dephasing.

In QET, Alice's measurement disturbs the quantum many-body system. That disturbance costs energy when the measured local observable does not commute with the Hamiltonian. We will study this more carefully after introducing Hamiltonians in Chapter 4.

2.16 Local and Global States: The First Glimpse

QET is about local actions on a larger quantum system. So we need to begin thinking about the difference between a global state and a local state.

A global state describes the whole system. A local state describes only part of it.

Suppose Alice has qubit A, and Bob has qubit B. The combined system is written using a tensor product Hilbert space,

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B.$$

We will study tensor products in detail in Chapter 3. For now, read ⊗ as "combine these systems into one larger system."

A famous two-qubit state is

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}},$$

where

$$|00\rangle = |0\rangle_A \otimes |0\rangle_B,$$

and

$$|11\rangle = |1\rangle_A \otimes |1\rangle_B.$$

This is an entangled state. Entanglement will be developed carefully in Chapter 3. For now, the important fact is this: the global state can contain correlations that are not visible from either subsystem alone.

If Alice measures her qubit in the \|0⟩,|1⟩\ basis, she obtains outcome 0 with probability frac12 and outcome 1 with probability frac12.

If Alice obtains 0, the global state becomes

$$|00\rangle.$$

If Alice obtains 1, the global state becomes

$$|11\rangle.$$

So Bob's conditional state becomes $|0\rangle$ when Alice gets 0, and $|1\rangle$ when Alice gets 1.

But if Bob does not know Alice's outcome, his local state is still a fifty-fifty mixture:

$$\rho_B = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|.$$

This example is one of the simplest ways to see why quantum measurement can change conditional states without allowing faster-than-light signaling. Alice's outcome matters, but Bob needs the classical message to know which conditional situation he is in. This no-signaling structure is a standard feature of local quantum operations and will become central in Chapter 5 (Nielsen and Chuang, 2010).

2.17 Selective Knowledge and Conditional States

Let us slow down because this point is directly connected to QET.

After Alice measures, there are two different descriptions we might use.

First, if we know Alice's outcome m, we use the conditional state

$$\rho_m = \frac{M_m \rho M_m^\dagger}{p(m)}.$$

Second, if we do not know Alice's outcome, we use the averaged state

$$\rho' = \sum_m M_m \rho M_m^\dagger.$$

These are not the same description.

In QET, Bob's action depends on the value of m. Alice sends m through a classical channel. Bob then applies a conditional operation, often a unitary operation U_m, chosen specifically for that outcome.

A unitary operator is an operator U satisfying

$$U^\dagger U = UU^\dagger = I.$$

Unitary operators describe reversible quantum evolutions. If a state ρ undergoes a unitary operation U, it becomes

$$\rho' = U\rho U^\dagger.$$

So the basic measurement-feedback pattern has the form

$$\rho \longrightarrow \rho_m \longrightarrow U_m \rho_m U_m^\dagger.$$

Averaged over outcomes, the final state is

$$\rho_{\text{final}} = \sum_m U_m M_m \rho M_m^\dagger U_m^\dagger.$$

This formula is already close to the mathematical structure of a QET protocol. Later, M_m will represent Alice's local measurement, and U_m will represent Bob's outcome-dependent local operation.

2.18 Commutators and Compatible Observables

Two observables do not always fit together peacefully.

The commutator of two operators A and B is

$$[A, B] = AB - BA.$$

If

$$[A, B] = 0,$$

then A and B commute. In many important cases, commuting observables can be simultaneously diagonalized and can have simultaneously definite values.

If

$$[A, B] \neq 0,$$

then the observables are incompatible in a quantum sense. Measuring one can disturb information about the other.

For example, Pauli X and Pauli Z do not commute:

$$XZ = -ZX,$$

so

$$[X, Z] = XZ - ZX = 2XZ \neq 0.$$

If a qubit is in an eigenstate of X, such as

$$|\psi_+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}},$$

then measuring Z generally destroys the definite X-property.

Commutators will matter for energy. If Alice measures an observable that does not commute with the relevant Hamiltonian terms, the measurement can change the energy expectation. This is one mathematical reason measurement can inject energy into a quantum system.

2.19 The Hamiltonian as an Observable

The Hamiltonian, usually written H, is the observable corresponding to energy. It also generates time evolution.

For a closed quantum system, if the state is pure, its time evolution is governed by the Schrödinger equation:

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle.$$

If H does not depend on time, the solution can be written as

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle.$$

The operator

$$U(t) = e^{-iHt/\hbar}$$

is unitary.

For a density matrix, closed-system time evolution is

$$\rho(t) = U(t)\rho(0)U(t)^\dagger.$$

The Hamiltonian will receive a full chapter later, but we introduce it here because QET is about energy changes. When a measurement changes the state from ρ to ρ', the energy expectation changes from

$$\mathrm{Tr}(\rho H)$$

to

$$\mathrm{Tr}(\rho' H).$$

The difference

$$\Delta E = \mathrm{Tr}(\rho' H) - \mathrm{Tr}(\rho H)$$

is the change in expected energy.

If $\Delta E>0$, energy has been injected into the system on average.

If $\Delta E<0$, energy has been extracted from the system on average.

In QET, Alice's measurement typically injects positive energy, while Bob's conditional operation can extract energy from his local region. The detailed balance depends on the Hamiltonian, the correlations in the initial state, and the chosen measurement and feedback operations.

2.20 A Tiny Energy Example

Consider a qubit with Hamiltonian

$$H = \frac{\hbar\omega}{2}(I - Z).$$

This Hamiltonian gives

$$H|0\rangle = 0,$$

and

$$H|1\rangle = \hbar\omega|1\rangle.$$

So |0⟩ is the ground state with energy 0, and |1⟩ is an excited state with energy hbarω.

Suppose the qubit begins in the ground state

$$\rho = |0\rangle\langle 0|.$$

Its energy expectation is

$$E = \mathrm{Tr}(\rho H) = 0.$$

Now measure X, whose projectors are

$$P_+ = |+\rangle\langle +|, \quad P_- = |-\rangle\langle -|,$$

where

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}},$$

and

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

Each outcome occurs with probability frac12. If we ignore the outcome, the final state is

$$\rho' = P_+ \rho P_+ + P_- \rho P_-.$$

One can calculate that

$$\rho' = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -| = \frac{I}{2}.$$

The final energy is

$$E' = \mathrm{Tr}\left(\frac{I}{2}H\right) = \frac{\hbar\omega}{2}.$$

So the measurement has increased the system's expected energy by

$$\Delta E = \frac{\hbar\omega}{2}.$$

Where did this energy come from?

It came from the measurement apparatus or the external agent implementing the measurement. Measurement is a physical interaction, not magic. This simple example prepares us for the more subtle many-body case: Alice's local measurement can inject energy into a larger quantum system even when the initial state is the ground state.

2.21 Why This Chapter Matters for QET

We can now state the measurement structure behind QET in the language of this chapter.

A typical QET protocol begins with a many-body system in a correlated low-energy state, often the ground state. Alice performs a local measurement described by operators

$$\{M_m^{(A)}\}.$$

The superscript A reminds us that the measurement acts only on Alice's region. The probability of outcome m is

$$p(m) = \mathrm{Tr}\left(M_m^{(A)\dagger} M_m^{(A)} \rho\right).$$

The conditional post-measurement state is

$$\rho_m = \frac{M_m^{(A)} \rho M_m^{(A)\dagger}}{p(m)}.$$

Alice sends m to Bob as classical information. Bob applies a local conditional operation

$$U_m^{(B)}.$$

The final averaged state is

$$\rho_{\mathrm{QET}} = \sum_m U_m^{(B)} M_m^{(A)} \rho M_m^{(A)\dagger} U_m^{(B)\dagger}.$$

This formula does not yet prove that

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