Chapter 5: The Dirac Field in Detail

In the previous chapter we quantized the free Dirac field and saw that its excitations are naturally interpreted as electrons and positrons. That construction used creation and annihilation operators, anticommutation relations, and the expansion of the field in positive- and negative-frequency spinor modes. We now slow down and study the spinor structure itself.

This chapter is deliberately technical. In QED calculations, most mistakes are not conceptual failures about photons or electrons; they are algebraic mistakes involving gamma matrices, spin sums, signs, normalizations, or polarization conventions. The purpose of this chapter is to make the Dirac field feel like a usable instrument.

We will use natural units,

$$\hbar=c=1,$$

and the mostly-minus metric convention,

$$\eta_{\mu\nu}=\mathrm{diag}(1,-1,-1,-1).$$

Repeated Lorentz indices are summed, and

$$a \cdot b = \eta_{\mu\nu} a^\mu b^\nu = a^0 b^0 - \mathbf{a} \cdot \mathbf{b}.$$

The central object is the Dirac equation,

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0,$$

introduced by Dirac in 1928 as a relativistic wave equation first order in both time and space derivatives (Dirac 1928). In modern QED, $\psi(x)$ is not merely a one-particle wavefunction. It is a spinor quantum field. But the algebra of its components is already encoded in the free equation above.

5.1 Why gamma matrices are needed

The relativistic energy-momentum relation is

$$p^2 = m^2, \qquad p^2 = p_\mu p^\mu = (p^0)^2 - \mathbf{p}^2.$$

The Klein-Gordon equation follows by replacing $p\mu$ with $i\partial\mu$:

$$(\partial^2 + m^2)\phi = 0.$$

It is second order in time. Dirac looked for an equation first order in derivatives,

$$(i\gamma^\mu \partial_\mu - m)\psi = 0,$$

whose square would imply the Klein-Gordon equation component by component. Apply the conjugate operator:

$$(i\gamma^\nu \partial_\nu + m)(i\gamma^\mu \partial_\mu - m)\psi = 0.$$

Since partial derivatives commute,

$$(i\gamma^\nu \partial_\nu)(i\gamma^\mu \partial_\mu) = -\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu.$$

For this to become $-\partial^2$, the symmetric part of **γ^nuγ^μ** must reproduce the metric. Thus the gamma matrices must satisfy the Clifford algebra

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{1}},$$

where

$$\{A, B\} = AB + BA$$

is the anticommutator.

This single relation is the foundation of almost all Dirac algebra. It implies

$$(\gamma^0)^2 = +\mathbf{1}, \qquad (\gamma^i)^2 = -\mathbf{1}, \qquad \gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu \quad (\mu \neq \nu).$$

The matrices γ^μ cannot be ordinary numbers, because ordinary numbers commute. They must act on a multi-component object. In four spacetime dimensions, the smallest faithful representation uses 4×4 matrices, so the Dirac field ψ has four complex components.

A useful shorthand is slash notation:

slashedaequiv γ^μ a_μ.

For example,

slashedp=γ^μ p_μ.

The Clifford algebra immediately gives

slashedpslashedp = p_μ p_nu γ^μγ^nu = p□1,

because the antisymmetric part of γ^μγ^nu drops out when contracted with the symmetric tensor pμ pnu.

Thus the momentum-space Dirac equation,

(slashedp-m)u(p)=0,

implies

$$(p^2 - m^2)u(p) = 0.$$

The Dirac equation is therefore a first-order square root of the relativistic mass-shell condition.

5.2 A concrete representation

The gamma matrices are not unique. A representation is a particular choice of matrices satisfying the Clifford algebra. Different representations are related by similarity transformations and describe the same physics.

One common choice is the Dirac representation:

$$\gamma^0 = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

where σ^i are the Pauli matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Pauli matrices satisfy

$$\sigma^i \sigma^j = \delta^{ij} \mathbf{1}_2 + i\epsilon^{ijk}\sigma^k.$$

Let us check one Clifford relation. For a spatial gamma matrix,

$$(\gamma^i)^2 = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = \begin{pmatrix} -\sigma^i\sigma^i & 0 \\ 0 & -\sigma^i\sigma^i \end{pmatrix} = -\mathbf{1}_4.$$

This matches $\eta^{ii}=-1$.

Another important representation is the chiral or Weyl representation:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix},$$

with

$$\sigma^\mu = (\mathbf{1}_2, \sigma^i), \qquad \bar{\sigma}^\mu = (\mathbf{1}_2, -\sigma^i).$$

The chiral representation is especially natural when studying massless fermions, weak interactions, and chirality. QED itself is vector-like, meaning that left- and right-chiral components couple to the photon with the same electric charge, but chiral notation remains extremely useful.

The important point is this:

$$\text{gamma matrices are a language, not an observable.}$$

Changing representation changes the components of ψ, but not measurable amplitudes, cross sections, or decay rates. Standard treatments of QED exploit this representation freedom heavily in practical calculations (Bjorken and Drell 1964; Peskin and Schroeder 1995).

5.3 The Dirac adjoint and Lorentz-invariant bilinears

The ordinary Hermitian product ψ^†ψ is positive, but it is not Lorentz invariant. The correct object for forming Lorentz-covariant quantities is the Dirac adjoint:

$$\boxed{\bar{\psi} \equiv \psi^\dagger \gamma^0.}$$

Then the scalar

$$\bar{\psi}\psi$$

is Lorentz invariant.

Why does γ^0 appear? The reason is that spinors do not transform under Lorentz transformations by unitary matrices in the ordinary finite-dimensional sense. Rotations can be represented unitarily, but boosts cannot be represented by finite-dimensional unitary matrices because the Lorentz group is noncompact. Instead, the Dirac adjoint is built to compensate for the spinor transformation law.

The Dirac Lagrangian density is

$$\mathcal{L}_D = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi.$$

Its mass term,

$$-m\bar{\psi}\psi,$$

is a Lorentz scalar. Its kinetic term,

$$i\bar{\psi}\gamma^\mu \partial_\mu \psi,$$

is also a Lorentz scalar because barψγ^μψ transforms as a Lorentz vector.

This vector,

$$\boxed{j^\mu = \bar{\psi}\gamma^\mu\psi,}$$

is the conserved Dirac current. For the free theory,

$$\partial_\mu j^\mu = 0.$$

In QED this current becomes the source for the electromagnetic field, and the interaction term will be

$$\mathcal{L}_{\text{int}} = -e\bar{\psi}\gamma^\mu A_\mu\psi.$$

Thus the same spinor bilinear that expresses probability-current conservation in the free Dirac theory becomes the electric current in the interacting theory.

5.4 Lorentz transformations and spinor generators

A Lorentz transformation is a linear transformation of spacetime coordinates that preserves the interval:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \qquad \eta_{\rho\sigma}\Lambda^\rho{}_\mu\Lambda^\sigma{}_\nu = \eta_{\mu\nu}.$$

Vectors transform with Lambda^μnu. Spinors transform with a different matrix, usually written S(Lambda):

$$\psi'(x') = S(\Lambda)\psi(x).$$

The Dirac equation should have the same form in every inertial frame. This requires

$$S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu.$$

This equation says that the gamma matrices mediate between the vector representation of the Lorentz group and the spinor representation.

For an infinitesimal Lorentz transformation,

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu,$$

where $\omega_{(\mu nu)}$=-$\omega_{(nu\mu)}$, the spinor transformation is

$$S(\Lambda) = 1 - \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu} + \mathcal{O}(\omega^2),$$

with spinor Lorentz generators

$$\boxed{S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu].}$$

It is also common to define

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu],$$

so that

$$S^{\mu\nu} = \frac{1}{2}\sigma^{\mu\nu}.$$

These generators satisfy the Lorentz algebra. This connects Dirac spinors to the representation theory of the Lorentz group. More deeply, relativistic particles are classified by unitary representations of the Poincaré group, as shown by Wigner; spin-frac12 particles correspond to particular representations whose field realization is given by spinors (Wigner 1939; Weinberg 1995).

Example: rotations

For a spatial rotation, the relevant generators are

$$J^i = \frac{1}{2}\epsilon^{ijk}S^{jk}.$$

In the Dirac representation one finds

$$J^i = \frac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}.$$

Thus a Dirac spinor contains two two-component spin-frac12 objects under rotations. This is why Pauli matrices keep reappearing inside Dirac spinor calculations.

Example: boosts

Boost generators are

$$K^i = S^{0i}.$$

Unlike rotation generators, boost generators are not Hermitian in the same finite-dimensional spinor representation. This is not a problem: S(Lambda) is not required to be unitary as a 4×4 matrix. Physical quantum states live in a Hilbert space with a unitary representation of the Poincaré group; the classical spinor components transform covariantly under finite-dimensional Lorentz representations.

This distinction prevents a common confusion. The spinor field transforms covariantly, while the Hilbert-space states transform unitarily.

5.5 The γ^5 matrix and chirality

Define

$$\boxed{\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3.}$$

With our metric convention,

$$(\gamma^5)^2 = \mathbf{1}, \qquad \{\gamma^5, \gamma^\mu\} = 0.$$

The eigenvalues of γ^5 are ± 1. This lets us define projection operators

$$P_R = \frac{1+\gamma^5}{2}, \qquad P_L = \frac{1-\gamma^5}{2}.$$

They satisfy

$$P_R^2 = P_R, \qquad P_L^2 = P_L, \qquad P_R P_L = 0, \qquad P_R + P_L = 1.$$

The projected spinors are

$$\psi_R = P_R\psi, \qquad \psi_L = P_L\psi.$$

These are called the right-chiral and left-chiral parts of the Dirac spinor. Chirality is therefore the eigenvalue of γ^5, or equivalently the decomposition of a Dirac spinor into its two irreducible Weyl components.

In the chiral representation,

$$\gamma^5 = \begin{pmatrix} -\mathbf{1}_2 & 0 \\ 0 & \mathbf{1}_2 \end{pmatrix},$$

so

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}.$$

The Dirac equation in chiral components is

$$i\bar{\sigma}^\mu \partial_\mu \psi_L = m\psi_R,$$

$$i\sigma^\mu \partial_\mu \psi_R = m\psi_L.$$

This pair of equations teaches an important lesson:

$$\text{the mass term couples left and right chirality.}$$

If m=0, the equations decouple:

$$i\bar{\sigma}^\mu \partial_\mu \psi_L = 0, \qquad i\sigma^\mu \partial_\mu \psi_R = 0.$$

Thus, for massless fermions, left- and right-chiral fields can propagate independently. This fact is central in the Standard Model, where weak interactions treat left- and right-chiral fermions differently. In QED, however, the photon coupling

$$-e\bar{\psi}\gamma^\mu A_\mu \psi$$

does not distinguish them.

5.6 Helicity and its relation to chirality

Helicity is the projection of spin along the direction of momentum. For a spin-frac12 particle, define

$$\hat{h} = \frac{\mathbf{\Sigma} \cdot \mathbf{p}}{2|\mathbf{p}|},$$

where

$$\Sigma^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}.$$

The eigenvalues of hath are

$$h = \pm \frac{1}{2}.$$

Many amplitude calculations instead use

$$\lambda = \frac{\mathbf{\Sigma} \cdot \mathbf{p}}{|\mathbf{p}|},$$

whose eigenvalues are $\lambda = \pm 1$. Both conventions describe the same physical information.

Helicity and chirality are often confused, so let us separate them carefully.

- Chirality is defined by γ^5. It is an intrinsic Lorentz-representation property of the spinor.
- Helicity is spin projected along momentum. It depends on the particle's momentum.

For a massive particle, helicity is frame-dependent. If the particle moves slower than light, an observer can boost past it and reverse the direction of its momentum while leaving its spin orientation unchanged. The helicity then changes sign.

For a massless particle, no observer can boost past it. Therefore helicity is Lorentz invariant under proper orthochronous Lorentz transformations. For massless positive-energy spinors, chirality and helicity coincide in the standard convention:

$$\psi_R \quad \leftrightarrow \quad \text{positive helicity},$$

$$\psi_L \quad \leftrightarrow \quad \text{negative helicity}.$$

Example: massless momentum along the z-axis

Let

$$p^\mu = (E, 0, 0, E).$$

For a two-component spinor xi,

$$\sigma^3 \xi_+ = +\xi_+, \qquad \sigma^3 \xi_- = -\xi_-.$$

The spinor xi_+ has spin aligned with the momentum direction, so it has positive helicity. The spinor xi_- has spin anti-aligned with the momentum direction, so it has negative helicity.

When m=0, the Weyl equations select definite relations between these two-component spinors and the left/right chiral components. This is why massless spinor-helicity methods, used widely in modern scattering-amplitude calculations, are so powerful. We will not develop the full spinor-helicity formalism here, but the conceptual seed is already present.

5.7 Plane-wave spinors u and v

The free Dirac field expansion uses two kinds of spinor solutions:

$$\psi(x) = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left[b_s(\mathbf{p}) u_s(p) e^{-ip\cdot x} + d_s^\dagger(\mathbf{p}) v_s(p) e^{ip\cdot x} \right].$$

Here

$$E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2},$$

and $p^0=E_p$. The spinors $u_s(p)$ satisfy

(slashedp-m)u□(p)=0,

while the spinors $v_s(p)$ satisfy

(slashedp+m)v□(p)=0.

The u-spinors describe positive-energy particle modes, while the v-spinors describe antiparticle modes after field quantization. This interpretation is one of the essential achievements of relativistic quantum field theory: the negative-frequency solutions are not discarded; they become creation operators for antiparticles.

A common normalization is

$$\bar{u}_s(p) u_r(p) = 2m\delta_{sr},$$

$$\bar{v}_s(p) v_r(p) = -2m\delta_{sr},$$

and

$$u_s^\dagger(p) u_r(p) = 2E_{\mathbf{p}}\delta_{sr}, \qquad v_s^\dagger(p) v_r(p) = 2E_{\mathbf{p}}\delta_{sr}.$$

Different books sometimes use different factors of 2m, 2E, or √(2E). The physics is unchanged if all external-state and field-normalization conventions are adjusted consistently. The conventions above are standard in many QED calculations (Peskin and Schroeder 1995).

Rest-frame spinors

In the rest frame,

$$p^\mu = (m, 0, 0, 0).$$

The Dirac equation for u becomes

$$(\gamma^0 - 1)u = 0.$$

In the Dirac representation this selects the upper two components:

$$u_s(\mathbf{0}) = \sqrt{2m} \begin{pmatrix} \xi_s \\ 0 \end{pmatrix},$$

where xi$_s$ is a two-component Pauli spinor, for example

$$\xi_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \xi_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Similarly, the v-spinors satisfy

$$(\gamma^0 + 1)v = 0,$$

so in the rest frame,

$$v_s(\mathbf{0}) = \sqrt{2m} \begin{pmatrix} 0 \\ \eta_s \end{pmatrix}.$$

Boosting these rest-frame spinors gives spinors at arbitrary momentum.

A useful explicit form for positive-energy spinors in the Dirac representation is

$$u_s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi_s \end{pmatrix}$$

in chiral notation, or equivalently one often writes in the Dirac representation

$$u_s(p) = \sqrt{E+m}\begin{pmatrix} \xi_s \\ \dfrac{\sigma\cdot\mathbf{p}}{E+m}\xi_s \end{pmatrix}.$$

This expression shows how the lower components are suppressed in the nonrelativistic limit:

$$\frac{|\mathbf{p}|}{E+m} \approx \frac{|\mathbf{p}|}{2m} \qquad (|\mathbf{p}| \ll m).$$

Thus the Dirac spinor reduces to a Pauli spinor plus small relativistic corrections. This is the beginning of the path toward nonrelativistic QED and atomic physics.

5.8 Spin sums and completeness

In scattering calculations we often do not observe the spin of every external fermion. We then sum over final spins and average over initial spins. Spin sums turn products of spinors into gamma-matrix expressions.

With the normalization above,

boxed ∑□ u□(p)baru□(p)=slashedp+m,

boxed ∑□ v□(p)barv□(p)=slashedp-m.

These are called completeness relations. They say that the positive-energy spinors span the positive-energy solution space, while the negative-energy spinors span the corresponding antiparticle solution space.

Why the first spin sum must have this form

The matrix

$$\sum_s u_s(p)\bar{u}_s(p)$$

must be a 4×4 matrix built from the available Lorentz-covariant objects p^μ, m, and γ^μ. It must also annihilate correctly under the Dirac operator:

(slashedp-m)u□(p)=0.

The combination slashedp+m has exactly the right property:

(slashedp-m)(slashedp+m) = p□-m□ = 0

on shell. Normalization fixes the coefficient.

Example: unpolarized electron line

Suppose an amplitude contains a factor

$$\bar{u}_{s'}(p')\Gamma u_s(p),$$

where Gamma is some product of gamma matrices. The squared amplitude contains

$$\left[\bar{u}_{s'}(p')\Gamma u_s(p)\right]\left[\bar{u}_{s'}(p')\Gamma u_s(p)\right]^* .$$

Using

$$\left(\bar{u}_{s'}(p')\Gamma u_s(p)\right)^* = \bar{u}_s(p)\bar{\Gamma} u_{s'}(p'),$$

where

$$\bar{\Gamma} = \gamma^0\Gamma^\dagger\gamma^0,$$

and summing over spins gives

∑□s,s'□ baru□s'□(p')Gamma u□(p) baru□(p)barGammau□s'□(p') = Tr≤ft[(slashedp'+m)Gamma (slashedp+m)barGamma].

This trace formula is one of the main reasons gamma-matrix technology is so efficient. It turns spin sums into algebra.

5.9 Gamma-matrix trace technology

The trace of a matrix is the sum of its diagonal entries. Traces are cyclic:

$$\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA) = \mathrm{Tr}(CAB).$$

In QED, traces appear whenever we square amplitudes and sum over unobserved fermion spins. The basic trace identities in four dimensions are

$$\mathrm{Tr}(\mathbf{1}) = 4,$$

$$\mathrm{Tr}(\gamma^\mu) = 0,$$

$$\mathrm{Tr}(\gamma^\mu \gamma^\nu) = 4\eta^{\mu\nu},$$

$$\mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho) = 0,$$

$$\mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4\left(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right).$$

More generally, the trace of an odd number of gamma matrices vanishes in four dimensions when no γ^5 is present.

For γ^5, with $\varepsilon^{0123}=+1$,

$$\mathrm{Tr}(\gamma^5) = 0,$$

$$\mathrm{Tr}(\gamma^5 \gamma^\mu \gamma^\nu) = 0,$$

$$\mathrm{Tr}(\gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = -4i\epsilon^{\mu\nu\rho\sigma}.$$

The sign of the last identity depends on the metric and γ^5 convention. The convention used here is consistent with

$$\eta_{\mu\nu} = (1, -1, -1, -1), \qquad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3, \qquad \epsilon^{0123} = +1.$$

Some useful contraction identities are

$$\gamma^\mu \gamma_\mu = 4,$$

$$\gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu,$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4\eta^{\nu\rho},$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\gamma^\sigma \gamma^\rho \gamma^\nu.$$

These identities follow directly from the Clifford algebra. They are not extra assumptions.

Example: a basic leptonic tensor

Consider the spin-summed current factor

L□μnu□ = Tr≤ft[(slashedp'+m)γ^μ (slashedp+m)γ^nu].

Expand:

L□μnu□ = Tr(slashedp'γ^μslashedpγ^nu) + m□Tr(γ^μγ^nu),

because the terms with three gamma matrices vanish. Now

Tr(slashedp'γ^μslashedpγ^nu) = p'_α p_β Tr(γ^αγ^μγ^βγ^nu).

Using the four-gamma trace,

$$L^{\mu\nu} = 4\left[p'^\mu p^\nu + p'^\nu p^\mu - (p' \cdot p)\eta^{\mu\nu} + m^2\eta^{\mu\nu}\right].$$

Thus

$$\boxed{L^{\mu\nu} = 4\left[p'^{\mu}p^{\nu} + p'^{\nu}p^{\mu} + (m^2 - p'\cdot p)\eta^{\mu\nu}\right].}$$

This kind of tensor will appear repeatedly in tree-level QED scattering.

5.10 The five bilinear covariants

A bilinear is an expression with one barψ, one ψ, and a gamma-matrix structure between them:

$$\bar{\psi}\Gamma\psi.$$

There are sixteen linearly independent 4×4 matrices that can be built from gamma matrices. They may be organized into five Lorentz types:

Type	Bilinear	Number of components
Scalar	barψψ	1
Pseudoscalar	barψiγ⁵ψ	1
Vector	barψγ^μψ	4
Axial vector	barψγ^μγ⁵ψ	4
Tensor	barψσ⁽μnu⁾ψ	6

The numbers add to

$$1 + 1 + 4 + 4 + 6 = 16,$$

which matches the dimension of the space of 4×4 complex matrices.

Let us define the less familiar names.

A scalar is unchanged under proper Lorentz transformations. The mass term is the central example:

$$\bar{\psi}\psi.$$

A pseudoscalar is unchanged under proper Lorentz transformations but changes sign under parity. A common Hermitian pseudoscalar is

$$\bar{\psi} i\gamma^5 \psi.$$

A vector transforms like x^μ. The electromagnetic current is

$$j^\mu = \bar{\psi}\gamma^\mu \psi.$$

An axial vector, also called a pseudovector, transforms like a vector under proper Lorentz transformations but differs under parity. The axial current is

$$j_5^\mu = \bar{\psi}\gamma^\mu \gamma^5 \psi.$$

A tensor has two antisymmetric Lorentz indices:

$$\bar{\psi}\sigma^{\mu\nu}\psi, \qquad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu].$$

Tensor bilinears appear, for example, in magnetic moment interactions. A Pauli-type term has the schematic form

$$\bar{\psi}\sigma^{\mu\nu}\psi F_{\mu\nu}.$$

In ordinary renormalizable QED for a pointlike electron, this is not a term in the fundamental dimension-four Lagrangian. But it appears as an effective operator at low energies and is closely related to anomalous magnetic moments.

The classification of bilinears is more than bookkeeping. It tells us which terms are allowed by Lorentz symmetry, parity, charge conjugation, and gauge invariance. This is the same logic that will later organize effective field theory.

5.11 Parity, charge conjugation, and time reversal

Discrete symmetries are transformations not continuously connected to the identity. The most important are:

- Parity P: spatial inversion,

$$(t, \mathbf{x}) \mapsto (t, -\mathbf{x}).$$

- Charge conjugation C: exchange of particles with antiparticles and reversal of additive charges.

- Time reversal

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