Chapter 3: Classical Fields, Symmetries, and Noether Structure

The previous chapter led us from relativistic wave equations to the need for fields. The Klein-Gordon and Dirac equations are not merely equations for one-particle wavefunctions; in relativistic physics they are better understood as field equations. This chapter builds the classical field-theoretic language in which QED will later be quantized.

The word classical here does not mean “nonrelativistic,” nor does it mean “macroscopic.” It means that the fields are still ordinary mathematical functions of spacetime rather than operators acting on a quantum Hilbert space. Before quantizing a theory, we first need to know:

- what its dynamical variables are,
- what equations they obey,
- what symmetries organize those equations,
- and what conserved quantities follow from those symmetries.

The central tool is the action principle. Instead of guessing equations of motion directly, we construct a scalar quantity S, called the action, from a Lagrangian density $\mathcal{L}$. The physical field configurations are those for which the action is stationary under small variations of the fields. This framework is especially powerful because Lorentz invariance and gauge symmetry can be imposed directly on $\mathcal{L}$, and because continuous symmetries lead systematically to conservation laws through Noether’s theorem, first formulated by Emmy Noether in 1918 (Noether 1918).

This chapter is the classical skeleton of QED. Later, when we quantize fields and compute scattering amplitudes, the same Lagrangian, currents, and stress-energy tensors will reappear in operator language and in Feynman rules.

3.1 Fields as spacetime degrees of freedom

A field assigns one or more numbers to every point of spacetime. In relativistic notation, a spacetime point is

$$x^\mu = (x^0, x^1, x^2, x^3) = (t, \mathbf{x}),$$

where we use units with

$$\hbar = c = 1.$$

Greek indices μ,nu,ρ,... run over 0,1,2,3. Latin spatial indices i,j,k,... run over 1,2,3. Throughout this chapter we use the mostly-minus Minkowski metric

$$\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1),$$

so that

$$x_\mu x^\mu = t^2 - \mathbf{x}^2.$$

The derivative with respect to spacetime is written

$$\partial_\mu = \frac{\partial}{\partial x^\mu}, \qquad \partial^\mu = \eta^{\mu\nu}\partial_\nu.$$

Thus

$$\partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \nabla^2 \equiv \Box,$$

where Box is the d'Alembertian.

Different physical fields transform differently under Lorentz transformations. Three types will be essential for QED.

A scalar field φ(x) assigns one number to each spacetime point and has no spacetime index. Under a Lorentz transformation x↦ x', a scalar field satisfies

$$\phi'(x') = \phi(x).$$

A spinor field ψ(x) has several complex components and transforms in a spinor representation of the Lorentz group. Spinors are needed to describe spin-frac12 matter such as electrons. We introduced the Dirac equation in Chapter 2; here we place it in Lagrangian form.

A vector field Aμ(x) carries a Lorentz index. The electromagnetic potential is a vector field, though it has an additional subtlety: not all components of Aμ are physically distinct, because electromagnetic theory has gauge redundancy.

The goal of a classical field theory is to specify equations of motion for these fields in a way compatible with special relativity. The Lagrangian formulation accomplishes this efficiently.

3.2 The action and the Euler-Lagrange equations for fields

In particle mechanics, one often starts from a Lagrangian L(q,dot q,t), where q(t) is a coordinate and dot q(t) is its time derivative. The action is

$$S[q] = \int dt\, L(q, \dot{q}, t).$$

The physical path is the one for which S is stationary under small variations of q(t) that vanish at the endpoints.

For fields, the dynamical variables are functions of spacetime. A field theory is described by a Lagrangian density

$$\mathcal{L}(\phi_A, \partial_\mu \phi_A, x),$$

where A labels the different field components. The action is

$$S[\phi_A] = \int d^4x\, \mathcal{L}(\phi_A, \partial_\mu \phi_A, x).$$

The field equations follow by varying the fields,

$$\phi_A(x) \to \phi_A(x) + \delta\phi_A(x),$$

and requiring δ S=0 for variations that vanish sufficiently rapidly at the boundary. The variation of the action is

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_A} \delta \phi_A + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_A)} \delta(\partial_\mu \phi_A) \right],$$

with a sum over repeated field labels A. Since

$$\delta(\partial_\mu \phi_A) = \partial_\mu(\delta \phi_A),$$

we integrate by parts:

$$\int d^4x \, \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_A)} \partial_\mu(\delta \phi_A) = -\int d^4x \, \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_A)} \right] \delta \phi_A,$$

dropping the boundary term. Therefore

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_A} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_A)} \right) \right] \delta \phi_A.$$

Because the variations δφ₍A₎ are arbitrary in the interior, the coefficient of each variation must vanish. The field-theoretic Euler-Lagrange equations are

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi_A} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_A)} \right) = 0.}$$

This equation is the classical engine behind scalar fields, spinor fields, gauge fields, and QED itself. Standard quantum field theory texts derive QED from precisely this Lagrangian starting point before quantization (Peskin and Schroeder 1995; Weinberg 1995).

3.3 The real scalar field: the simplest relativistic field theory

The simplest Lorentz-invariant field theory is a real scalar field φ(x). “Real” means that φ^=φ. A standard free scalar Lagrangian density is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2.$$

The first term is the kinetic term, involving derivatives of the field. The second term is the mass term. The parameter m will become the particle mass after quantization.

Applying the Euler-Lagrange equation,

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi,$$

and

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi.$$

Thus

$$-m^2\phi - \partial_\mu\partial^\mu\phi = 0,$$

or

$$\boxed{(\Box + m^2)\phi = 0.}$$

This is the Klein-Gordon equation.

The Lagrangian formulation also makes interactions easy to include. For example,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

gives

$$(\Box + m^2)\phi + \frac{\lambda}{3!}\phi^3 = 0.$$

The coefficient 4! is conventional. It is chosen because, in perturbation theory, it cancels combinatorial factors associated with four identical fields at an interaction vertex.

This scalar theory is not QED, but it is pedagogically important. It shows how a Lorentz-invariant Lagrangian produces a relativistic wave equation, and it introduces the language of kinetic terms, mass terms, and interaction terms that will be used throughout the rest of the book.

3.4 Canonical momentum and Hamiltonian density

In mechanics, the momentum conjugate to q(t) is

$$p = \frac{\partial L}{\partial \dot{q}}.$$

For a field φ(t,x), the canonical momentum density conjugate to φ is

$$\pi(t, \mathbf{x}) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}.$$

For the free real scalar field,

$$\mathcal{L} = \frac{1}{2}(\partial_0 \phi)^2 - \frac{1}{2}(\nabla \phi)^2 - \frac{1}{2}m^2\phi^2,$$

so

$$\pi = \partial_0 \phi.$$

The Hamiltonian density is the field-theoretic Legendre transform,

$$\mathcal{H} = \pi\, \partial_0 \phi - \mathcal{L}.$$

For the free scalar field,

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2.$$

The total Hamiltonian is

$$H = \int d^3x\, \mathcal{H}.$$

This expression is positive for the free real scalar field. After quantization, H will become the energy operator. In Chapter 4 we will promote φ and π to operators and impose canonical commutation relations.

The Hamiltonian viewpoint is useful for quantization, but the Lagrangian viewpoint is usually more transparent for Lorentz invariance and gauge symmetry. QED calculations often move between both languages.

3.5 Complex scalar fields and internal phase symmetry

A real scalar field has one real component. A complex scalar field has two real components packaged as

$$\phi(x) = \frac{1}{\sqrt{2}}\left(\phi_1(x) + i\phi_2(x)\right).$$

A standard free complex scalar Lagrangian is

$$\mathcal{L} = \partial_\mu\phi^*\,\partial^\mu\phi - m^2\phi^*\phi.$$

In varying the action, φ and φ^ are treated as independent variables. The Euler-Lagrange equations give

$$(\Box + m^2)\phi = 0, \qquad (\Box + m^2)\phi^* = 0.$$

This theory has a continuous internal symmetry:

$$\phi(x) \to e^{-i\alpha}\phi(x), \qquad \phi^*(x) \to e^{i\alpha}\phi^*(x),$$

where α is a constant real number.

The word internal means that the transformation changes the field values but not the spacetime point x. A rotation in ordinary space changes directions in space. An internal phase rotation changes the complex phase of the field at each point without moving the point.

For infinitesimal α,

$$\delta\phi = -i\alpha\phi, \qquad \delta\phi^* = i\alpha\phi^*.$$

This symmetry will lead to a conserved current. In QED, the analogous phase symmetry becomes the organizing principle behind electric charge.

3.6 Continuous symmetries and Noether’s theorem

A symmetry of an action is a transformation of the fields, and possibly of spacetime coordinates, that leaves the action unchanged. Continuous symmetries depend smoothly on one or more parameters. Examples include:

- time translations t→ t+a,
- spatial translations x→ x+a,
- rotations,
- Lorentz boosts,
- internal phase rotations $\varphi \to e^{(-i\alpha)}\varphi$.

Noether’s theorem states that every continuous global symmetry of the action gives a conserved current (Noether 1918). A current is a four-vector j^μ(x) whose conservation law is

$$\boxed{\partial_\mu j^\mu = 0.}$$

Writing j^μ=(ρ,j), this becomes

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{j} = 0.$$

This is a local conservation law. It says that the density ρ can change in a small region only if current flows into or out of that region.

The associated conserved charge is

$$Q = \int d^3x\, j^0(t, \mathbf{x}).$$

Assuming the spatial current vanishes sufficiently rapidly at infinity,

$$\frac{dQ}{dt} = \int d^3x\, \partial_0 j^0 = -\int d^3x\, \nabla \cdot \mathbf{j} = -\oint_\infty d\mathbf{S} \cdot \mathbf{j} = 0.$$

Thus Q is time independent.

For an internal transformation

$$\phi_A \to \phi_A + \delta\phi_A,$$

with

$$\delta\phi_A = \epsilon\, \Delta\phi_A,$$

where ε is a constant infinitesimal parameter, suppose the Lagrangian changes only by a total derivative:

$$\delta\mathcal{L} = \epsilon\, \partial_\mu K^\mu.$$

Then the Noether current is

$$\boxed{j^\mu = \sum_A \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_A)} \Delta\phi_A - K^\mu.}$$

On the equations of motion,

$$\partial_\mu j^\mu = 0.$$

The phrase on the equations of motion is often shortened to on shell. A statement is on shell if it holds for field configurations satisfying the Euler-Lagrange equations. A statement is off shell if it holds for arbitrary field configurations.

This distinction will matter later. Ward identities, gauge fixing, and renormalization all involve careful separation of identities that hold because of equations of motion from identities that follow algebraically from symmetry.

3.7 Example: the conserved current of a complex scalar field

Return to the complex scalar Lagrangian

$$\mathcal{L} = \partial_\mu \phi^* \, \partial^\mu \phi - m^2 \phi^* \phi.$$

Under the infinitesimal global phase rotation

$$\delta\phi = -i\alpha\phi, \qquad \delta\phi^* = i\alpha\phi^*,$$

we identify

$$\Delta\phi = -i\phi, \qquad \Delta\phi^* = i\phi^*.$$

The Lagrangian is exactly invariant, so K^μ=0. The required derivatives are

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi^*, \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} = \partial^\mu \phi.$$

Therefore

$$j^\mu = (\partial^\mu \phi^*)(-i\phi) + (\partial^\mu \phi)(i\phi^*),$$

or

$$\boxed{j^\mu = i\left(\phi^*\partial^\mu\phi - \phi\,\partial^\mu\phi^*\right).}$$

This is often written using the bidirectional derivative notation

$$\phi^*\overleftrightarrow{\partial^\mu}\phi = \phi^*\partial^\mu\phi - (\partial^\mu\phi^*)\phi,$$

so that

$$j^\mu = i\phi^*\overleftrightarrow{\partial^\mu}\phi.$$

The conserved charge is

$$Q = \int d^3x\, i\left(\phi^*\partial^0\phi - \phi\,\partial^0\phi^*\right).$$

After quantization, this charge counts particles minus antiparticles for the complex scalar field. The important lesson for QED is structural: a global phase symmetry produces a conserved charge current.

3.8 The Dirac field in Lagrangian form

The Dirac equation from Chapter 2 was

$$(i\gamma^\mu\partial_\mu - m)\psi = 0.$$

Here ψ(x) is a four-component spinor field, and the gamma matrices satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbf{1}.$$

The Dirac adjoint is defined by

$$\bar{\psi} = \psi^\dagger\gamma^0.$$

The reason for this definition is that barψψ transforms as a Lorentz scalar, whereas ψ^†ψ does not. More generally, barψγ^μψ transforms as a Lorentz vector. These transformation properties are central to constructing Lorentz-invariant spinor Lagrangians, as treated in standard relativistic field theory references (Weinberg 1995; Peskin and Schroeder 1995).

The Dirac Lagrangian density is

$$\boxed{\mathcal{L}_D = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi.}$$

In varying the action, ψ and barψ are treated as independent fields. Variation with respect to barψ gives

$$(i\gamma^\mu \partial_\mu - m)\psi = 0.$$

Variation with respect to ψ gives the adjoint Dirac equation,

$$i(\partial_\mu \bar{\psi})\gamma^\mu + m\bar{\psi} = 0.$$

It is useful to see explicitly that the Dirac Lagrangian produces a conserved current. Consider the global phase transformation

$$\psi \to e^{-i\alpha}\psi, \qquad \bar{\psi} \to \bar{\psi}e^{i\alpha}.$$

For infinitesimal α,

$$\delta\psi = -i\alpha\psi, \qquad \delta\bar{\psi} = i\alpha\bar{\psi}.$$

The derivative-dependent part of the Lagrangian is

$$i\bar{\psi}\gamma^\mu \partial_\mu \psi.$$

Thus

$$\frac{\partial \mathcal{L}_D}{\partial(\partial_\mu \psi)} = i\bar{\psi}\gamma^\mu.$$

There is no derivative of barψ in this unsymmetrized form of the Lagrangian. The Noether current is therefore

$$j^\mu = i\bar{\psi}\gamma^\mu(-i\psi),$$

so

$$\boxed{j^\mu = \bar{\psi}\gamma^\mu\psi.}$$

This is the familiar Dirac current. Its time component is

$$j^0 = \bar{\psi}\gamma^0\psi = \psi^\dagger\psi.$$

In one-particle Dirac theory, j^0 was interpreted as a probability density. In field theory, the same current becomes a charge current. After quantization, the associated charge counts electrons and positrons with opposite signs.

For QED, this current will couple to the electromagnetic field.

3.9 The electromagnetic field as a classical gauge field

The electromagnetic field can be described by a four-potential

$$A_\mu(x) = (A_0, \mathbf{A}),$$

from which the electromagnetic field strength tensor is built:

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.}$$

The tensor F(μnu) is antisymmetric:

$$F_{\mu\nu} = -F_{\nu\mu}.$$

Its components contain the electric and magnetic fields. Up to conventional sign choices associated with index placement,

$$F_{\mu\nu} \quad \longleftrightarrow \quad \mathbf{E}, \mathbf{B}.$$

The free electromagnetic Lagrangian is

$$\boxed{\mathcal{L}_{\text{EM}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}.}$$

This compact expression is Lorentz invariant and yields the source-free Maxwell equations. The covariant Lagrangian formulation of electromagnetism is standard in classical field theory treatments of Maxwell theory (Jackson 1999).

Let us derive the equations. Since

$$\delta F_{\mu\nu} = \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu,$$

the variation is

$$\delta \mathcal{L}_{\text{EM}} = -\frac{1}{2} F^{\mu\nu} \delta F_{\mu\nu}.$$

Using antisymmetry of F⁽μnu⁾,

$$\delta \mathcal{L}_{\text{EM}} = -F^{\mu\nu} \partial_\mu \delta A_\nu.$$

Integrating by parts gives

$$\delta S_{\text{EM}} = \int d^4x \, (\partial_\mu F^{\mu\nu}) \delta A_\nu.$$

Therefore the Euler-Lagrange equations are

$$\boxed{\partial_\mu F^{\mu\nu} = 0.}$$

With an external conserved current j^μ, one writes

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu.$$

The field equation becomes

$$\boxed{\partial_\mu F^{\mu\nu} = j^\nu.}$$

The other pair of Maxwell equations is not obtained by varying the action. It follows from the definition of F(μnu). Since F=dA in differential-form language, it automatically satisfies

$$\boxed{\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0.}$$

This is the Bianchi identity. In vector notation it contains

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0.$$

Thus the potential Aμ packages Maxwell's equations into a form adapted to relativity and, later, quantization.

3.10 Gauge redundancy

The electromagnetic potential is not unique. If

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x),$$

where Lambda(x) is any sufficiently smooth scalar function, then

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = F_{\mu\nu} + \partial_\mu\partial_\nu\Lambda - \partial_\nu\partial_\mu\Lambda = F_{\mu\nu}.$$

Because partial derivatives commute,

$$\partial_\mu\partial_\nu\Lambda = \partial_\nu\partial_\mu\Lambda.$$

So the physical electric and magnetic fields are unchanged.

This is called gauge redundancy. It is not a symmetry in quite the same sense as a spatial rotation or a global phase rotation. A rotation maps one physical configuration into another physically equivalent but usually distinct configuration. A gauge transformation changes the mathematical representative Aμ without changing the physical electromagnetic field F(μnu).

The word "redundancy" is important. It means that the potential Aμ contains more components than the physical photon has. A massless spin-1 particle has two physical polarization states, but Aμ has four components. Gauge symmetry and constraints remove the unphysical components. This will become central in Chapter 6 and Chapter 13.

Gauge invariance also constrains allowed couplings. For an external current,

$$S_{\text{int}} = -\int d^4x\, j_\mu A^\mu.$$

Under Aμ→ Aμ+∂μLambda,

$$\delta S_{\text{int}} = -\int d^4x\, j^\mu \partial_\mu \Lambda.$$

Integrating by parts,

$$\delta S_{\text{int}} = \int d^4x\, (\partial_\mu j^\mu)\Lambda,$$

up to a boundary term. Since Lambda(x) is arbitrary, gauge invariance requires

$$\boxed{\partial_\mu j^\mu = 0.}$$

Thus electromagnetic gauge invariance is compatible only with conserved charge. In QED this connection becomes deeper: local phase redundancy of the matter field and gauge redundancy of Aμ become one unified structure.

3.11 Global versus local symmetries

A global symmetry uses the same transformation parameter everywhere in spacetime. For example,

$$\psi(x) \to e^{-i\alpha}\psi(x)$$

with constant α is a global phase symmetry.

A local symmetry allows the parameter to depend on spacetime:

$$\psi(x) \to e^{-i\alpha(x)}\psi(x).$$

At first, this seems like a small change. It is not. If α depends on x, then

$$\partial_\mu\psi \to \partial_\mu(e^{-i\alpha(x)}\psi) = e^{-i\alpha(x)}\left[\partial_\mu\psi - i(\partial_\mu\alpha)\psi\right].$$

The derivative produces an extra term involving $\partial_\mu\alpha$. Therefore the free Dirac Lagrangian

$$\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

is invariant under constant phase rotations but not under local phase rotations.

The cure is to introduce a new field A_μ and replace the ordinary derivative by a covariant derivative. In QED, this will be

$$D_\mu = \partial_\mu + ieA_\mu$$

or, depending on charge-sign convention,

$$D_\mu = \partial_\mu - iqA_\mu.$$

The covariant derivative is designed to transform like the field itself under local phase transformations. We will derive this carefully in Chapter 7. For now, the important point is conceptual:

> Making a global phase symmetry local forces the introduction of a gauge field.

This is the organizing principle behind QED. It is also one of the central organizing principles of the Standard Model.

Local gauge symmetry differs from global symmetry in its Noether meaning. A global continuous symmetry gives a conserved charge. A local gauge redundancy gives identities among equations of motion and removes unphysical degrees of freedom. This distinction is associated with Noether's first and second theorems (Noether 1918). In modern field theory language, global symmetries act on physical states, while gauge redundancies relate different descriptions of the same physical state.

3.12 Spacetime translations and the stress-energy tensor

So far our examples involved internal transformations. Now consider spacetime translations,

$$x^\mu \to x^\mu + a^\mu,$$

where a^μ is constant. If the Lagrangian has no explicit dependence on x, the action is invariant under translations. Noether's theorem then gives conservation of energy and momentum.

The associated current is the stress-energy tensor. For fields $\phi_{(A)}$, the canonical stress-energy tensor is

$$\boxed{T^\mu{}_\nu = \sum_A \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_A)} \partial_\nu \phi_A - \delta^\mu{}_\nu \mathcal{L}.}$$

Translation invariance implies

$$\boxed{\partial_\mu T^\mu{}_\nu = 0.}$$

The conserved four-momentum is

$$P_\nu = \int d^3x \, T^0{}_\nu.$$

In raised-index form, \(P

Document information

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Project	Quantum Electrodynamics
Document	Document 1.7
Author	terry.mart
Verifier	Not verified
Downloaded	July 04, 2026 22:51 KST
Status	Working
Document link	https://theorytrace.com/projects/quantum-electrodynamics/documents/chapter-3-classical-fields-symmetries-and-noether-structure/