Chapter 2: Quantum Mechanics Needed for Control

Quantum control is not possible without quantum mechanics, but we do not need the whole subject at once. We need the parts that answer three practical questions:

1. What is the state of the system?
2. How does the state change in time?
3. How do controls such as laser, microwave, magnetic, or electric pulses change that motion?

This chapter builds the minimum foundation for those questions. We will use the standard state-vector language of quantum mechanics, often called Dirac notation, and the standard postulates for measurement and time evolution as presented in undergraduate quantum mechanics and quantum information texts (Griffiths and Schroeter, 2018; Nielsen and Chuang, 2010).

The goal is not to memorize symbols. The goal is to learn how quantum systems are represented so that, in later chapters, a pulse, a field, or a feedback rule can be understood as a way of steering a state.

2.1 Quantum states: the object we control

In classical mechanics, the state of a simple particle might be described by its position and velocity. If you know those values, Newton’s laws tell you how the particle will move.

In quantum mechanics, the state is different. A quantum state is a mathematical object that lets us predict the probabilities of possible measurement outcomes. For a pure state, we represent the state by a vector written as

$$|\psi\rangle.$$

This symbol is called a ket. It is a vector in a complex vector space called a Hilbert space. For this book, you can think of a Hilbert space as the allowed mathematical space of states, with a rule for measuring lengths and angles between vectors. The full mathematical definition is more advanced, but the key point is simple: quantum states behave like vectors, and we can add them and multiply them by complex numbers.

A complex number has the form

$$a + ib,$$

where a and b are real numbers and $i^2=-1$. Complex numbers are essential in quantum mechanics because they carry both magnitude and phase. Phase will become central in control, because changing phases is one of the main ways pulses steer quantum systems.

A state vector is usually normalized, meaning its total probability is 1. In symbols,

$$\langle \psi | \psi \rangle = 1.$$

Here $\langle \psi|$ is called a bra, and $\langle \psi|\psi\rangle$ is an inner product. For now, you can read this as “the length squared of the state vector.”

The important idea is this:

> Quantum control tries to move the state vector from where it is now to where we want it to be.

For example, in a quantum computer, we may start with a qubit in the state $|0\rangle$ and want to move it to another state such as

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}}.$$

In an atomic clock, we may want to prepare atoms in a state whose phase evolves very regularly. In a spin sensor, we may want to prepare a spin so that a tiny magnetic field changes its phase in a measurable way.

2.2 Basis states: naming the directions

To describe a vector, we choose a set of reference directions. In ordinary two-dimensional geometry, we might use the x-axis and y-axis. In quantum mechanics, we choose basis states.

A basis is a set of states from which other states can be built. For a two-level system, the most common basis is

$$|0\rangle, \quad |1\rangle.$$

These are often called computational basis states in quantum information (Nielsen and Chuang, 2010). A general pure state of a two-level system can be written as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,$$

where α and β are complex numbers. The normalization condition is

$$|\alpha|^2 + |\beta|^2 = 1.$$

The numbers α and β are called probability amplitudes. They are not probabilities themselves. Their squared magnitudes give probabilities when we measure in the |0⟩, |1⟩ basis.

For example, suppose

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle.$$

Then a measurement in the |0⟩, |1⟩ basis gives

$$P(0) = \left|\frac{\sqrt{3}}{2}\right|^2 = \frac{3}{4},$$

and

$$P(1) = \left|\frac{1}{2}\right|^2 = \frac{1}{4}.$$

So the system has a 75% chance of being found in |0⟩ and a 25% chance of being found in |1⟩.

This does not mean the system was secretly in either |0⟩ or |1⟩ before we measured. The state above is a genuine quantum superposition.

2.3 Superposition: more than “either/or”

A superposition is a linear combination of possible basis states. The state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

is a superposition when both α and β are nonzero.

This idea is often described too casually as “the system is in two states at once.” That phrase can be useful as a first image, but it is not precise. A better statement is:

> A superposition is a state whose measurement outcomes are described by amplitudes that can interfere with each other.

The word interference is important. In quantum mechanics, amplitudes can add or cancel before probabilities are calculated. This is different from ordinary probability.

Consider the two states

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}},$$

and

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

If we measure either state in the |0⟩, |1⟩ basis, both give

$$P(0) = \frac{1}{2}, \quad P(1) = \frac{1}{2}.$$

So in that measurement basis, they look the same. But they are not the same state. The minus sign in |-⟩ is a relative phase. If we later apply another operation, |+⟩ and |-⟩ can behave very differently.

This is one reason quantum control is subtle. It is not enough to control probabilities. We often need to control phases.

2.4 Global phase and relative phase

Quantum states can include phases. A phase is an angle-like quantity that appears in complex numbers such as

$$e^{i\phi} = \cos\phi + i\sin\phi.$$

If a whole state is multiplied by the same phase factor,

$$|\psi\rangle \to e^{i\gamma}|\psi\rangle,$$

then all physical predictions remain the same. This is called a global phase. The global phase is not directly observable in ordinary quantum measurements.

But the phase between components of a superposition is observable. This is called relative phase.

For example,

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

and

$$\frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

differ by a relative phase of π between the |0⟩ and |1⟩ parts. That relative phase affects interference and is therefore physically meaningful.

In quantum control, many pulses are designed to change relative phases accurately. A small unwanted phase error can turn a correct quantum gate into an incorrect one, or reduce the sensitivity of a quantum sensor.

2.5 Measurement: how quantum information becomes an outcome

A quantum state gives probabilities, not guaranteed outcomes. The rule connecting states to measurement results is called the Born rule. In standard quantum mechanics, if a system is in the state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,$$

then a measurement in the $|0\rangle$, $|1\rangle$ basis gives outcome 0 with probability

$$|\alpha|^2,$$

and outcome 1 with probability

$$|\beta|^2.$$

This rule is one of the basic postulates of quantum mechanics (Griffiths and Schroeter, 2018; Nielsen and Chuang, 2010).

After a simple ideal measurement, the state changes to match the observed result. If the outcome is 0, the state becomes $|0\rangle$. If the outcome is 1, the state becomes $|1\rangle$. This change is often called state collapse in introductory language.

For example, let

$$|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}.$$

A measurement in the $|0\rangle$, $|1\rangle$ basis gives 0 with probability 1/2 and 1 with probability 1/2. If the result is 0, then immediately after the measurement the state is $|0\rangle$.

This matters for control because measurement is not passive. In classical physics, we often imagine that measuring a system can be made arbitrarily gentle. In quantum mechanics, measurement generally affects the system. Later, in feedback control, we will use measurement results to decide what control action to apply next.

2.6 Operators: actions on quantum states

An operator is a mathematical object that acts on a state and produces another state. If hatA is an operator, then

$$\hat{A}|\psi\rangle$$

means “apply hatA to the state |ψ⟩.”

In finite-dimensional quantum systems, operators can be represented by matrices. States can be represented by column vectors. For example, we can represent

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then the operator

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

acts as

$$X|0\rangle = |1\rangle,$$

and

$$X|1\rangle = |0\rangle.$$

This operator is called the Pauli X operator. In quantum computing, it is also called a bit-flip operation because it swaps |0⟩ and |1⟩.

Two other important Pauli operators are

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

and

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Z operator leaves |0⟩ unchanged but changes the sign of |1⟩:

$$Z|0\rangle = |0\rangle,$$

$$Z|1\rangle = -|1\rangle.$$

So Z changes relative phase. This is why phase control is as important as population control.

2.7 Observables and eigenvalues

Some operators represent measurable quantities. These are called observables. In standard quantum mechanics, observables are represented by Hermitian operators, whose eigenvalues are real numbers and can represent possible measurement outcomes (Griffiths and Schroeter, 2018).

An eigenstate of an operator is a state that is not changed in direction by that operator. If

$$\hat{A}|a\rangle = a|a\rangle,$$

then |a⟩ is an eigenstate of hatA, and a is the corresponding eigenvalue.

For example,

$$Z|0\rangle = |0\rangle.$$

So |0⟩ is an eigenstate of Z with eigenvalue +1.

Also,

$$Z|1\rangle = -|1\rangle.$$

So |1⟩ is an eigenstate of Z with eigenvalue -1.

If we measure the observable Z, the possible outcomes are +1 and -1. A system in |0⟩ gives +1 with certainty. A system in |1⟩ gives -1 with certainty.

But a system in

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

gives +1 with probability 1/2 and -1 with probability 1/2.

This example teaches an important control lesson:

> A state can be certain for one measurement and uncertain for another.

The state |+⟩ is uncertain for a Z measurement, but it is an eigenstate of the X operator:

$$X|+\rangle = |+\rangle.$$

So |+⟩ gives a definite outcome for an X measurement.

2.8 The Hamiltonian: the generator of motion

The most important operator for dynamics is the Hamiltonian, usually written as H. The Hamiltonian represents the energy structure and interactions of the quantum system. It determines how the state changes in time through the Schrödinger equation,

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle.$$

Here:

- i is the imaginary unit,
- hbar is the reduced Planck constant,
- t is time,
- |ψ(t)⟩ is the state at time t,
- H is the Hamiltonian.

This equation is the central dynamical law for closed quantum systems in nonrelativistic quantum mechanics (Griffiths and Schroeter, 2018).

If the Hamiltonian does not change with time, the solution can be written as

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle,$$

where

$$U(t) = e^{-iHt/\hbar}.$$

The operator U(t) is called a unitary time-evolution operator.

A unitary operator preserves the length of the state vector. This is necessary because total probability must remain 1. In quantum control, many control goals are stated as desired unitary operations: for example, “apply a gate that rotates this qubit from |0⟩ to |+⟩.”

2.9 Energy eigenstates and phase evolution

Suppose the Hamiltonian has an eigenstate |E⟩ with energy E:

$$H|E\rangle = E|E\rangle.$$

If the system starts in this energy eigenstate, then time evolution gives

$$|\psi(t)\rangle = e^{-iEt/\hbar}|E\rangle.$$

This is only a global phase if the state contains just one energy eigenstate. A global phase alone does not change measurement probabilities.

But if the state is a superposition of two energy eigenstates,

$$|\psi(0)\rangle = c_1|E_1\rangle + c_2|E_2\rangle,$$

then after time t,

$$|\psi(t)\rangle = c_1 e^{-iE_1 t/\hbar}|E_1\rangle + c_2 e^{-iE_2 t/\hbar}|E_2\rangle.$$

Now the relative phase changes at a rate determined by the energy difference

$$E_2 - E_1.$$

This is one of the deepest reasons quantum control works. If we can control energies or couplings, we can control how phases accumulate.

For example, in an atomic clock, atoms are prepared in a superposition of two energy states. The relative phase between those states evolves at a frequency connected to the energy difference. Measuring that phase evolution allows extremely precise timekeeping. Later chapters will explain how control pulses prepare, manipulate, and read out such phase information.

2.10 Two-level systems: the simplest useful quantum system

Many quantum technologies are built from systems where only two states matter. These are called two-level systems.

A two-level system may be:

- two energy levels of an atom,
- two spin states of an electron,
- two states of a superconducting circuit,
- two polarization states of a photon,
- two hyperfine states of an ion or neutral atom.

Mathematically, we often call the two basis states

$$|0\rangle, \quad |1\rangle.$$

A general pure state is

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,$$

with

$$|\alpha|^2 + |\beta|^2 = 1.$$

Two-level systems are important because they are simple enough to visualize but rich enough to show superposition, interference, phase, measurement disturbance, and coherent control.

In quantum computing, a controllable two-level system is called a qubit. The word qubit means “quantum bit.” A classical bit can be 0 or 1. A qubit can be in a superposition of |0⟩ and |1⟩. However, when measured in the computational basis, it gives a classical result: either 0 or 1.

This difference is central to quantum technology. A qubit is not useful merely because it can be in a superposition. It is useful because we can control superpositions, phases, entanglement with other qubits, and measurement outcomes in carefully designed sequences (Nielsen and Chuang, 2010).

2.11 Spin: a physical example of a two-level system

One of the most important physical examples of a two-level system is a spin-(1)/(2) particle, such as an electron.

Spin is an intrinsic form of angular momentum. It is not exactly the same as a tiny ball spinning in space. Instead, it is a quantum property with measurement outcomes that follow quantum rules.

For a spin-(1)/(2) system, a measurement of spin along a chosen axis gives only two possible outcomes:

$$+\frac{\hbar}{2} \quad \text{or} \quad -\frac{\hbar}{2}.$$

If we choose the z-axis, we often write the two states as

$$|\uparrow_z\rangle, \quad |\downarrow_z\rangle.$$

In quantum information language, we may identify

$$|0\rangle = |\uparrow_z\rangle,$$

$$|1\rangle = |\downarrow_z\rangle.$$

A magnetic field can affect the spin because spin has a magnetic moment. Very roughly, the spin behaves like a tiny magnetic object that can interact with an external magnetic field. More precisely, the Hamiltonian contains terms that depend on the magnetic field and the spin operators. This is the basis of many control methods in nuclear magnetic resonance, electron spin resonance, and spin-based quantum sensing.

For example, a constant magnetic field along the z-axis can make the relative phase between |uparrow(z)⟩ and |downarrow(z)⟩ change over time. A carefully timed oscillating field can rotate the spin from one direction to another. Chapter 3 will develop this idea using pulses and resonance.

2.12 The Bloch sphere: seeing a qubit

A two-level pure state can be visualized using the Bloch sphere. This is a sphere where each point on the surface represents one possible pure qubit state, ignoring global phase. The Bloch-sphere representation is standard in quantum information because it gives a geometric picture of single-qubit states and operations (Nielsen and Chuang, 2010).

Any pure qubit state can be written as

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle,$$

where

$$0 \leq \theta \leq \pi,$$

and

$$0 \leq \phi < 2\pi.$$

The angles θ and φ locate a point on the sphere.

The north pole is

$$|0\rangle.$$

The south pole is

$$|1\rangle.$$

A point on the equator with φ=0 is

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}.$$

A point on the equator with φ=π is

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

Another equator state is

$$|+i\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}.$$

This corresponds to a different relative phase.

The Bloch sphere is not a physical ball surrounding the particle. It is a map of possible states. Still, it is extremely useful for control.

A control pulse can often be pictured as a rotation of the state around an axis on the Bloch sphere. For example:

- A pulse corresponding to an X-rotation moves the state around the x-axis.
- A pulse corresponding to a Y-rotation moves the state around the y-axis.
- Free evolution under a Z-type Hamiltonian rotates the phase around the z-axis.

This geometric picture will become one of our main tools.

2.13 Pauli operators as rotation axes

The Pauli operators X, Y, and Z are not only observables. They also help describe rotations of a qubit.

A rotation around the x-axis by an angle θ is represented by

$$R_x(\theta) = e^{-i\theta X/2}.$$

Similarly,

$$R_y(\theta) = e^{-i\theta Y/2},$$

and

$$R_z(\theta) = e^{-i\theta Z/2}.$$

The factor of 1/2 is not a typo. It appears because qubit states have spinor geometry: a 2π rotation changes the sign of the state vector, although physical measurement probabilities remain unchanged because the sign is a global phase.

As an example, consider applying a π-rotation around the x-axis to $|0\rangle$:

$$R_x(\pi)|0\rangle = e^{-i\pi X/2}|0\rangle.$$

Using the identity

$$e^{-i\theta X/2} = \cos\left(\frac{\theta}{2}\right) I - i \sin\left(\frac{\theta}{2}\right) X,$$

we get

$$R_x(\pi) = -iX.$$

Therefore,

$$R_x(\pi)|0\rangle = -iX|0\rangle = -i|1\rangle.$$

The final state differs from |1⟩ only by the global phase -i, so physically it is the |1⟩ state.

This is why a π-pulse is often described as flipping a two-level system.

2.14 Unitary control: steering without losing probability

A closed quantum system evolves unitarily. This means its state changes by a unitary operator U:

$$|\psi_{\text{final}}\rangle = U|\psi_{\text{initial}}\rangle.$$

In quantum control, we often design fields so that the actual unitary evolution is close to a desired target unitary.

For example, suppose we want to create the transformation

$$|0\rangle \to |+\rangle.$$

One way is to apply a rotation around the y-axis:

$$R_y\left(\frac{\pi}{2}\right)|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}.$$

So a control task might be:

> Design a pulse that implements R(y)(π/2) accurately on the physical qubit.

This sentence contains the spirit of quantum control. We have a mathematical target, a physical system, and a control field that must make the system follow the desired transformation.

In real experiments, the implemented operation may not be perfect. The pulse may be slightly too long, the field may be slightly too weak, the system may interact with its environment, or nearby unwanted energy levels may be affected. Later chapters will describe how controllability, decoherence, optimal control, and feedback deal with these problems.

2.15 Time-dependent Hamiltonians: where control enters

So far, we have often imagined that the Hamiltonian is fixed. But control usually means changing the Hamiltonian in time.

A useful model is

$$H(t) = H_0 + \sum_k u_k(t) H_k.$$

Here:

- H_0 is the natural Hamiltonian of the system when no control is applied.
- H_k are control Hamiltonians describing how the system responds to available fields.
- $u_k(t)$ are time-dependent control functions, such as pulse amplitudes or voltages.

For example, a qubit might have a natural energy splitting described by

$$H_0 = \frac{\hbar \omega_0}{2} Z.$$

A microwave field might add a control term approximately proportional to X:

$$H_{\text{control}}(t) = \hbar \Omega(t) X.$$

Then the total Hamiltonian is

$$H(t) = \frac{\hbar \omega_0}{2} Z + \hbar \Omega(t) X.$$

The function $\Omega(t)$ describes the strength of the applied drive over time. By shaping $\Omega(t)$, we shape the motion of the qubit.

This is the mathematical bridge from quantum mechanics to quantum control:

> Controls are time-dependent choices that modify the Hamiltonian and therefore modify the unitary evolution.

Chapter 3 will explore this bridge in detail through resonance, Rabi oscillations, rotating frames, and pulse design.

2.16 What you should understand before moving on

At this point, the most important ideas are these.

A quantum state is a vector that gives probabilities for measurement outcomes. A two-level pure state has the form

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,$$

with

$$|\alpha|^2 + |\beta|^2 = 1.$$

The numbers α and β are amplitudes. Their squared magnitudes give probabilities in the chosen measurement basis, but their relative phase also matters.

Operators act on states. Some operators represent observables. The Hamiltonian is the operator that generates time evolution through the Schrödinger equation.

Unitary evolution preserves total probability. For a time-independent Hamiltonian,

$$U(t) = e^{-iHt/\hbar}.$$

For a qubit, the Pauli operators X, Y, and Z provide a compact language for measurement axes, phase changes, and rotations. The Bloch sphere turns this language into a geometric picture.

Most importantly, quantum control begins when we make the Hamiltonian time-dependent:

$$H(t) = H_0 + \sum_k u_k(t) H_k.$$

The functions $u_k(t)$ are the control knobs. They may represent laser intensity, microwave amplitude, magnetic-field strength, voltage, phase, or frequency. By designing those functions carefully, we steer the state.

That is the foundation. We are now ready to study how pulses move quantum systems in time.

References

Griffiths, D. J., and Schroeter, D. F. (2018). Introduction to Quantum Mechanics (3rd ed.). Cambridge University Press.

Nielsen, M. A., and Chuang, I. L. (2010). Quantum Computation and Quantum Information: 10th Anniversary Edition. Cambridge University Press.

Document information

Chapter 2: Quantum Mechanics Needed for Control

Project	Quantum Control in Action
Document	Document 1.6
Author	mujirin
Verifier	Not verified
Downloaded	July 03, 2026 18:17 KST
Status	Working
Document link	https://theorytrace.com/projects/quantum-control-in-action/documents/chapter-2-quantum-mechanics-needed-for-control/