Chapter 5: Projective Measurements

In Chapter 4, we learned that a quantum state is a mathematical object that produces probabilities. A pure state is represented by a unit vector lvert ψ⟩, and a general state is represented by a density operator ρ, meaning

$$\rho \geq 0, \qquad \mathrm{Tr}(\rho) = 1.$$

We now study the first standard model of quantum measurement: projective measurement.

Projective measurements are the most direct quantum analogue of measuring an observable such as spin, energy, or position in an idealized finite-dimensional system. They are built from orthogonal projections, which are operators that represent yes/no questions about whether a state lies in a certain subspace.

This chapter has two goals.

First, we will understand projective measurements rigorously:

- what an orthogonal projector is,
- what a projection-valued measure is,
- how the Born rule assigns probabilities,
- how spectral decompositions produce measurements,
- and why ideal projective measurements are repeatable.

Second, we will understand why projective measurements are not the full story. In quantum information, many natural measurement tasks require generalized measurements, later called POVMs. Naimark dilation will eventually show that these generalized measurements can still be understood through projective measurements on a larger Hilbert space.

Projective measurements and POVMs are standard parts of the mathematical formulation of quantum information theory (Nielsen and Chuang, 2010; Watrous, 2018).

5.1 The basic question: what does a measurement return?

A measurement is an experiment with possible outcomes.

For example, if we measure a classical coin, the possible outcomes may be

$$\{\text{heads}, \text{tails}\}.$$

If we measure a qubit in the computational basis, the possible outcomes are usually written

$$\{0, 1\}.$$

A quantum measurement must answer two mathematical questions:

1. Probability question:

Given a state ρ, what is the probability of each outcome?

2. State-update question:

After an outcome is observed, what is the new state?

Projective measurements answer both questions using orthogonal projectors.

Before defining the full measurement, we need to understand projectors carefully.

5.2 Orthogonal projectors

Let □ be a finite-dimensional complex Hilbert space.

An orthogonal projector is a linear operator P:□→□ satisfying

$$P^2 = P$$

and

$$P^* = P.$$

The condition $P^2=P$ says that applying P twice has the same effect as applying it once. The condition P^ast=P says that P is self-adjoint, so it behaves geometrically like a projection onto a subspace.

The subspace onto which P projects is its range:

$$\operatorname{ran}(P) = \{P|\psi\rangle : |\psi\rangle \in \mathcal{H}\}.$$

If lvert ψ⟩ is already in ran(P), then

$$P|\psi\rangle = |\psi\rangle.$$

If lvert ψ⟩ is orthogonal to ran(P), then

$$P|\psi\rangle = 0.$$

So an orthogonal projector keeps the part of a vector lying in a chosen subspace and removes the orthogonal part.

5.3 Example: projecting onto one basis vector

Consider the qubit Hilbert space

$$\mathcal{H} = \mathbb{C}^2$$

with computational basis

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The projector onto the one-dimensional subspace spanned by lvert 0⟩ is

$$P_0 = |0\rangle\langle 0|.$$

As a matrix,

$$P_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Similarly,

$$P_1 = |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Check that P₀ is a projector:

$$P_0^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = P_0.$$

Also,

$$P_0^* = P_0,$$

because the matrix is equal to its conjugate transpose.

Now take an arbitrary qubit vector

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Then

$$P_0|\psi\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix} = \alpha|0\rangle.$$

The projector P₀ extracts the lvert 0⟩-component of the state.

5.4 Projectors as yes/no questions

A projector represents a yes/no question.

For example,

$$P_0 = |0\rangle\langle 0|$$

represents the question:

> Is the qubit in the subspace spanned by lvert 0⟩?

The answer “yes” corresponds to P_0. The answer “no” corresponds to the complementary projector

$$I - P_0 = P_1.$$

More generally, if P is an orthogonal projector, then

$$I - P$$

is also an orthogonal projector. It projects onto the orthogonal complement of ran(P).

So the pair

$$\{P, I - P\}$$

represents a two-outcome projective measurement.

5.5 The Born rule for one projector

Suppose a quantum system is in a pure state lvert ψ⟩, where

$$\langle \psi | \psi \rangle = 1.$$

If P is an orthogonal projector, the Born rule says that the probability of obtaining the “yes” outcome associated with P is

$$p(\text{yes}) = \langle \psi | P | \psi \rangle.$$

Because P is positive semidefinite, this number is nonnegative. Because $P \leq I$, this number is at most 1. Thus it is a valid probability.

For a density operator ρ, the Born rule becomes

$$p(\text{yes}) = \mathrm{Tr}(P\rho).$$

Equivalently, since the trace is cyclic in finite dimensions,

$$\mathrm{Tr}(P\rho) = \mathrm{Tr}(\rho P).$$

This trace formula is the form most useful in quantum information theory, because it works for both pure and mixed states (Nielsen and Chuang, 2010; Watrous, 2018).

5.6 Example: measuring a qubit in the computational basis

Let

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,$$

where

$$|\alpha|^2 + |\beta|^2 = 1.$$

The computational-basis measurement has projectors

$$P_0 = |0\rangle\langle 0|, \qquad P_1 = |1\rangle\langle 1|.$$

The probability of outcome 0 is

$$p(0) = \langle\psi|P_0|\psi\rangle.$$

Since

$$P_0|\psi\rangle = \alpha|0\rangle,$$

we get

$$p(0) = |\alpha|^2.$$

Similarly,

$$p(1) = |\beta|^2.$$

So the familiar rule

$$\alpha|0\rangle + \beta|1\rangle \quad \rightsquigarrow \quad \begin{cases} 0 & \text{with probability } |\alpha|^2, \\ 1 & \text{with probability } |\beta|^2 \end{cases}$$

is an example of the Born rule for projectors.

5.7 Projection-valued measures in finite dimensions

We now define projective measurements with more than two possible outcomes.

Let X be a finite set of outcomes. For example,

$$X = \{0, 1\}$$

or

$$X = \{a, b, c\}.$$

A projection-valued measure, abbreviated PVM, on X is a family of orthogonal projectors

$$\{P_x\}_{x \in X}$$

such that

$$P_x P_y = 0 \quad \text{whenever } x \neq y,$$

and

$$\sum_{x \in X} P_x = I.$$

The condition

$$P_x P_y = 0$$

means that the subspaces for different outcomes are orthogonal. The condition

$$\sum_{x \in X} P_x = I$$

means that the measurement accounts for all possibilities.

In a finite-dimensional setting, this is the simplest form of a projection-valued measure. More generally, one can assign projectors not only to individual outcomes but also to sets of outcomes. If S⊆ X, define

$$P(S) = \sum_{x \in S} P_x.$$

Then P(S) is the projector corresponding to the event “the outcome lies in S.”

For example, if

$$X = \{1, 2, 3\}$$

and

$$S = \{1, 3\},$$

then

$$P(S) = P_1 + P_3.$$

The word “measure” appears because this construction resembles a probability measure, except that it assigns projectors first and probabilities only after a quantum state is supplied.

5.8 The Born rule for a PVM

Let

$$\{P_x\}_{x \in X}$$

be a PVM on ▯, and let ρ be a density operator.

The probability of outcome x is

$$p(x) = \mathrm{Tr}(P_x \rho).$$

This produces a genuine probability distribution.

First, each probability is nonnegative. Since $P_x \geq 0$ and $\rho \geq 0$,

$$\mathrm{Tr}(P_x \rho) \geq 0.$$

Second, the probabilities sum to 1:

$$\sum_{x \in X} p(x) = \sum_{x \in X} \mathrm{Tr}(P_x \rho).$$

By linearity of the trace,

$$\sum_{x \in X} \mathrm{Tr}(P_x \rho) = \mathrm{Tr}\left(\sum_{x \in X} P_x \rho\right).$$

Since

$$\sum_{x \in X} P_x = I,$$

we get

$$\mathrm{Tr}\left(\sum_{x\in X} P_x\rho\right) = \mathrm{Tr}(I\rho) = \mathrm{Tr}(\rho) = 1.$$

Therefore

$$p(x) = \mathrm{Tr}(P_x\rho)$$

is a valid probability distribution.

This calculation is important because it shows why projectors and normalization fit naturally with probability.

5.9 Pure-state form of the PVM Born rule

If the system is in a pure state

$$\rho = |\psi\rangle\langle\psi|,$$

then

$$p(x) = \mathrm{Tr}\big(P_x|\psi\rangle\langle\psi|\big).$$

Using the identity

$$\mathrm{Tr}(A|\psi\rangle\langle\psi|) = \langle\psi|A|\psi\rangle,$$

we obtain

$$p(x) = \langle\psi|P_x|\psi\rangle.$$

Because P_x is an orthogonal projector,

$$\langle\psi|P_x|\psi\rangle = \|P_x|\psi\rangle\|^2.$$

So, for a pure state, the probability of outcome x is the squared length of the component of lvert ψ⟩ inside the subspace corresponding to x.

This gives a clear geometric interpretation:

> A projective measurement decomposes the Hilbert space into mutually orthogonal outcome subspaces. The probability of an outcome is the squared length of the state’s projection into that outcome subspace.

5.10 Example: a three-dimensional projective measurement

Let

$$\mathcal{H} = \mathbb{C}^3$$

with orthonormal basis

$$|e_1\rangle, \quad |e_2\rangle, \quad |e_3\rangle.$$

Define

$$P_1 = |e_1\rangle\langle e_1|,$$

and

$$P_2 = |e_2\rangle\langle e_2| + |e_3\rangle\langle e_3|.$$

Then

$$P_1 P_2 = 0,$$

and

$$P_1 + P_2 = I.$$

Thus

$$\{P_1, P_2\}$$

is a two-outcome PVM.

Notice that P_2 has rank 2. It does not distinguish between lvert e_2⟩ and lvert e_3⟩. It only asks whether the state lies in the two-dimensional subspace

$$\operatorname{span}\{|e_2\rangle, |e_3\rangle\}.$$

Now let

$$|\psi\rangle = \alpha|e_1\rangle + \beta|e_2\rangle + \gamma|e_3\rangle,$$

where

$$|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1.$$

Then

$$p(1) = \|P_1|\psi\rangle\|^2 = |\alpha|^2,$$

and

$$p(2) = \|P_2|\psi\rangle\|^2 = |\beta|^2 + |\gamma|^2.$$

This example shows that projective measurements can be coarse-grained. A measurement can combine several basis directions into one outcome.

5.11 Fine-grained and coarse-grained projective measurements

A projective measurement is called fine-grained when every nonzero projector has rank 1. In that case, each outcome corresponds to a one-dimensional subspace.

For example, in mathbbC³, the PVM

$$\{|e_1\rangle\langle e_1|, |e_2\rangle\langle e_2|, |e_3\rangle\langle e_3|\}$$

is fine-grained.

A projective measurement is coarse-grained when at least one projector has rank greater than 1. It combines several orthogonal directions into one outcome.

For example,

$$\{|e_1\rangle\langle e_1|, |e_2\rangle\langle e_2| + |e_3\rangle\langle e_3|\}$$

is coarse-grained.

Coarse-graining loses information. It tells us that the state lies in a larger subspace but does not identify the exact direction inside that subspace.

5.12 Observables and spectral decompositions

In many physics texts, measurements are introduced through observables.

An observable in finite-dimensional quantum theory is represented by a self-adjoint operator A. Self-adjoint means

$$A^* = A.$$

Self-adjoint operators are important because their eigenvalues are real, so they can represent possible numerical measurement outcomes.

The finite-dimensional spectral theorem says that every self-adjoint operator A can be written as

$$A = \sum_{\lambda} \lambda P_\lambda,$$

where:

- λ ranges over the distinct eigenvalues of A,

- Pλ is the orthogonal projector onto the eigenspace of A with eigenvalue λ,
- the projectors satisfy

$$P_\lambda P_\mu = 0 \quad \text{if } \lambda \neq \mu,$$

and

$$\sum_\lambda P_\lambda = I.$$

Thus the spectral decomposition of A automatically gives a PVM:

$$\{P_\lambda\}_\lambda.$$

Measuring the observable A means performing this PVM. The probability of obtaining eigenvalue λ is

$$p(\lambda) = \mathrm{Tr}(P_\lambda \rho).$$

The expectation value of the measurement is

$$\mathbb{E}[A] = \sum_\lambda \lambda\, p(\lambda).$$

Using the Born rule,

$$\mathbb{E}[A] = \sum_\lambda \lambda \; \mathrm{Tr}(P_\lambda \rho).$$

By linearity of trace,

$$\mathbb{E}[A] = \mathrm{Tr}\left(\left(\sum_\lambda \lambda P_\lambda\right) \rho\right).$$

Since

$$A = \sum_{\lambda} \lambda P_{\lambda},$$

we get

$$\mathbb{E}[A] = \mathrm{Tr}(A\rho).$$

This is the usual expectation-value formula for observables in quantum theory (Nielsen and Chuang, 2010).

5.13 Example: measuring the Pauli Z observable

For a qubit, the Pauli Z operator is

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In Dirac notation,

$$Z = 1 \cdot |0\rangle\langle 0| + (-1) \cdot |1\rangle\langle 1|.$$

So the spectral projectors are

$$P_{+1} = |0\rangle\langle 0|, \qquad P_{-1} = |1\rangle\langle 1|.$$

Measuring Z gives outcome +1 with probability

$$p(+1) = \mathrm{Tr}(P_{+1}\rho),$$

and outcome -1 with probability

$$p(-1) = \mathrm{Tr}(P_{-1}\rho).$$

For a pure state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,$$

we have

$$p(+1) = |\alpha|^2, \qquad p(-1) = |\beta|^2.$$

The expectation value is

$$\mathbb{E}[Z] = (+1)|\alpha|^2 + (-1)|\beta|^2 = |\alpha|^2 - |\beta|^2.$$

5.14 Example: measuring the Pauli X observable

The Pauli X operator is

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It has eigenvectors

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle),$$

and

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle),$$

with eigenvalues +1 and -1, respectively.

Thus

$$X = 1 \cdot |+\rangle\langle+| + (-1) \cdot |-\rangle\langle-|.$$

The projective measurement associated with X is

$$\{|+\rangle\langle+|, |-\rangle\langle-|\}.$$

Now suppose the system is in state lvert 0⟩. Then

$$p(+) = |\langle+|0\rangle|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2},$$

and

$$p(-) = |\langle-|0\rangle|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}.$$

So a qubit that is definite in the Z-basis is not definite in the X-basis.

This is one of the first signs that quantum measurement is not simply revealing a pre-existing classical value.

5.15 State update after a projective measurement

So far we have discussed the probabilities of measurement outcomes. We now discuss what happens to the state after a measurement outcome is observed.

Let

$$\{P_x\}_{x\in X}$$

be a PVM, and suppose the initial state is ρ.

If outcome x occurs and

$$p(x) = \mathrm{Tr}(P_x\rho) > 0,$$

then the standard projective measurement update rule gives the new state

$$\rho_x = \frac{P_x \rho P_x}{\mathrm{Tr}(P_x \rho)}.$$

This is called the selective post-measurement state, because it is conditioned on the selected outcome x.

Let us check that ρ_x is a density operator.

First, it is positive semidefinite. For any vector lvert φ⟩,

$$\langle \phi | P_x \rho P_x | \phi \rangle = \langle P_x \phi | \rho | P_x \phi \rangle \geq 0,$$

because $\rho \geq 0$.

Second, it has trace 1:

$$\mathrm{Tr}(\rho_x) = \frac{\mathrm{Tr}(P_x \rho P_x)}{\mathrm{Tr}(P_x \rho)}.$$

Using $P_x^2 = P_x$ and cyclicity of trace,

$$\mathrm{Tr}(P_x \rho P_x) = \mathrm{Tr}(P_x^2 \rho) = \mathrm{Tr}(P_x \rho).$$

Therefore

$$\mathrm{Tr}(\rho_x) = 1.$$

So the update rule really produces a valid quantum state.

5.16 Pure-state update

Suppose the initial state is pure:

$$\rho = |\psi\rangle\langle\psi|.$$

If outcome x occurs, then

$$\rho_x = \frac{P_x|\psi\rangle\langle\psi|P_x}{\langle\psi|P_x|\psi\rangle}.$$

When Pₓlvert ψ⟩≠ 0, define

$$|\psi_x\rangle = \frac{P_x|\psi\rangle}{\|P_x|\psi\rangle\|}.$$

Then

$$\rho_x = |\psi_x\rangle\langle\psi_x|.$$

So for pure states, projective measurement has a simple geometric meaning:

1. Project the state vector into the outcome subspace.
2. Normalize the resulting vector.

For example, if

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

and we measure in the computational basis, then outcome 0 gives

$$P_0|\psi\rangle = \alpha|0\rangle.$$

After normalization, the state becomes

$$|0\rangle.$$

Similarly, outcome 1 gives the post-measurement state

$$|1\rangle.$$

5.17 Nonselective measurement

Sometimes we perform a measurement but do not learn, record, or condition on the outcome. This is called a nonselective measurement.

If the initial state is ρ, then after a projective measurement $P_x\backslash(x\in X)$ without conditioning on the outcome, the final state is

$$\rho' = \sum_{x\in X} P_x \rho P_x.$$

This formula is the average of the selective post-measurement states weighted by their probabilities.

Indeed, outcome x occurs with probability

$$p(x) = \mathrm{Tr}(P_x\rho),$$

and the conditional state is

$$\rho_x = \frac{P_x\rho P_x}{p(x)}.$$

Therefore the average state is

$$\sum_x p(x)\rho_x = \sum_x p(x)\frac{P_x\rho P_x}{p(x)} = \sum_x P_x\rho P_x.$$

A nonselective projective measurement can destroy coherence between different outcome subspaces.

For example, take

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle).$$

Then

$$\rho = |+\rangle\langle +| = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}.$$

Measure in the computational basis with

$$P_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The nonselective post-measurement state is

$$\rho' = P_0 \rho P_0 + P_1 \rho P_1.$$

Compute:

$$P_0 \rho P_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$P_1 \rho P_1 = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus

$$\rho' = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2} I.$$

The off-diagonal terms have disappeared. Those off-diagonal terms represented coherence between lvert 0⟩ and lvert 1⟩.

5.18 Repeatability of projective measurements

A key feature of ideal projective measurements is repeatability.

Repeatability means:

> If a projective measurement gives outcome x, and we immediately perform the same projective measurement again, then the second measurement gives the same outcome x with probability 1.

Let us prove this.

After outcome x, the state is

$$\rho_x = \frac{P_x \rho P_x}{\mathrm{Tr}(P_x \rho)}.$$

Now the probability of obtaining outcome y in a second measurement is

$$p(y|x) = \mathrm{Tr}(P_y \rho_x).$$

Substitute the formula for ρ_x:

$$p(y|x) = \frac{\mathrm{Tr}(P_y P_x \rho P_x)}{\mathrm{Tr}(P_x \rho)}.$$

If $y \neq x$, then

$$P_y P_x = 0,$$

so

$$p(y|x) = 0.$$

If $y = x$, then

$$p(x|x) = \frac{\mathrm{Tr}(P_x P_x \rho P_x)}{\mathrm{Tr}(P_x \rho)}.$$

Using $P_x^2 = P_x$,

$$p(x|x) = \frac{\mathrm{Tr}(P_x \rho P_x)}{\mathrm{Tr}(P_x \rho)}.$$

Again,

$$\mathrm{Tr}(P_x \rho P_x) = \mathrm{Tr}(P_x \rho),$$

so

$$p(x|x) = 1.$$

Thus projective measurement is repeatable in the sense of repeating the same outcome.

There is one subtle point. If P_x has rank greater than 1, then outcome x only identifies a subspace, not a unique vector. Repeating the measurement gives the same outcome, but it does not necessarily reveal a unique basis state inside that subspace.

5.19 Degeneracy

An observable A has a degenerate eigenvalue if its eigenspace has dimension greater than 1.

For example, consider the operator on mathbbC³

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The eigenvalue 1 has eigenspace

$$\mathrm{span}\{|e_1\rangle\},$$

while the eigenvalue 2 has eigenspace

$$\mathrm{span}\{|e_2\rangle, |e_3\rangle\}.$$

Thus

$$A = 1 \cdot P_1 + 2 \cdot P_2,$$

where

$$P_1 = |e_1\rangle\langle e_1|$$

and

$$P_2 = |e_2\rangle\langle e_2| + |e_3\rangle\langle e_3|.$$

The measurement of A has two outcomes, 1 and 2. If outcome 2 occurs, the measurement tells us that the state lies in the two-dimensional eigenspace for eigenvalue 2, but it does not distinguish lvert e_2⟩ from lvert e_3⟩.

Degeneracy is important because projective measurement outcomes need not correspond to one-dimensional rays. They may correspond to higher-dimensional subspaces.

5.20 Projective measurements as decompositions of the identity

A PVM

$$\{P_x\}_{x\in X}$$

is exactly an orthogonal decomposition of the identity operator.

The equation

$$\sum_{x\in X} P_x = I$$

means that every vector lvert ψ⟩∈▯ decomposes as

$$|\psi\rangle = \sum_{x\in X} P_x|\psi\rangle.$$

The orthogonality condition

$$P_xP_y = 0 \quad (x \neq y)$$

means that these components are mutually orthogonal.

Therefore

$$\||\psi\rangle\|^2 = \sum_{x \in X} \|P_x|\psi\rangle\|^2.$$

For a unit vector, this becomes

$$1 = \sum_{x \in X} \|P_x|\psi\rangle\|^2.$$

But

$$\|P_x|\psi\rangle\|^2 = p(x).$$

So the Born probabilities are precisely the squared lengths of the orthogonal pieces of the state vector.

This is one of the cleanest geometric pictures in finite-dimensional quantum mechanics.

5.21 Why projective measurements are limited

Projective measurements are mathematically elegant, but they are not general enough for quantum information.

There are several limitations.

5.21.1 Too few outcome directions in small dimension

In a d-dimensional Hilbert space, a PVM can have at most d nonzero rank-one outcomes. More generally, the nonzero projectors of a PVM must project onto mutually orthogonal subspaces whose dimensions add up to d.

For a qubit, d=2. A projective measurement can have at most two nonzero rank-one outcomes.

But quantum information often uses measurements on a qubit with three or more meaningful outcomes. For example, in state discrimination problems, one may want a measurement with outcomes corresponding to several possible signal states. Such measurements are naturally described by POVMs rather than PVMs (Nielsen and Chuang, 2010; Watrous, 2018).

A three-outcome qubit measurement cannot be a rank-one PVM

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