

Chapter 4: Measurement and Probability

In the previous chapter, we described quantum states using vectors. A single qubit can be written as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,$$

where α and β are complex amplitudes satisfying

$$|\alpha|^2 + |\beta|^2 = 1.$$

For n qubits, a state can be written as a superposition over computational basis states:

$$|\psi\rangle = \sum_{x \in \{0,1\}^n} \alpha_x |x\rangle,$$

with normalization condition

$$\sum_{x \in \{0,1\}^n} |\alpha_x|^2 = 1.$$

So far, we have treated quantum states as mathematical objects. But a computation must eventually produce an answer. That means we need to ask:

> How do amplitudes become observable outcomes?

The answer is measurement.

Measurement is the bridge between the quantum state and the classical information we can actually read. It is also one of the main reasons quantum computing is subtle. A quantum state may contain many amplitudes, but measurement does not reveal all of them. Instead, measurement produces a random classical outcome according to probabilities determined by the amplitudes, and it changes the state afterward.

This chapter explains that process carefully.

We will learn:

- how probabilities are computed from amplitudes,

- what measurement in the computational basis means,
- what state collapse means,
- how measurement works for several qubits,
- why relative phase matters even when probabilities look the same,
- how repeated measurements estimate probabilities,
- and why measurement limits access to quantum information.

These ideas are part of the standard circuit model of quantum computation (Nielsen and Chuang, 2010; Mermin, 2007). They will be essential for Grover's algorithm because Grover's algorithm does not directly print the answer. It reshapes amplitudes so that, when we finally measure, a correct answer is likely to appear.

4.1 Measurement turns amplitudes into probabilities

A quantum amplitude is not itself a probability. It may be negative or complex, so it cannot directly represent an ordinary probability.

For example, the number

$$\frac{i}{\sqrt{2}}$$

is a valid amplitude, but it is not a probability. Probabilities must be real numbers between 0 and 1.

The rule that converts amplitudes into probabilities is called the Born rule. In the computational basis, the Born rule says:

> The probability of observing a basis state is the squared magnitude of its amplitude.

For a single qubit

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,$$

measurement in the computational basis gives

$$P(0) = |\alpha|^2, \quad P(1) = |\beta|^2.$$

The normalization condition

$$|\alpha|^2 + |\beta|^2 = 1$$

guarantees that the probabilities add to 1, as probabilities should.

The squared magnitude of a complex number $z = a + bi$ is

$$|z|^2 = a^2 + b^2.$$

Equivalently,

$$|z|^2 = z^* z,$$

where z^* is the complex conjugate of z .

So if an amplitude is complex, we do not square it in the ordinary algebraic sense. We take its squared magnitude.

4.2 Example: measuring a simple superposition

Consider the qubit

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle.$$

The amplitudes are

$$\alpha = \frac{1}{\sqrt{2}}, \quad \beta = \frac{1}{\sqrt{2}}.$$

Therefore,

$$P(0) = \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2},$$

and

$$P(1) = \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}.$$

So measuring this qubit gives 0 half the time and 1 half the time.

This state is often written as

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}.$$

It is called the plus state. It is an equal superposition of $|0\rangle$ and $|1\rangle$.

The phrase “equal superposition” means that the magnitudes of the amplitudes are equal. It does not mean that the qubit secretly has both classical values in a directly readable way. If we measure it once, we see only one classical outcome: either 0 or 1.

4.3 Example: complex amplitudes

Now consider

$$|\psi\rangle = \frac{3}{5}|0\rangle + \frac{4i}{5}|1\rangle.$$

First, check normalization:

$$\left| \frac{3}{5} \right|^2 + \left| \frac{4i}{5} \right|^2 = \frac{9}{25} + \frac{16}{25} = 1.$$

So this is a valid quantum state.

The measurement probabilities are

$$P(0) = \left| \frac{3}{5} \right|^2 = \frac{9}{25},$$

and

$$P(1) = \left| \frac{4i}{5} \right|^2 = \frac{16}{25}.$$

The imaginary unit i affects the amplitude, but the probability depends on squared magnitude. Since

$$|i|^2 = 1,$$

the factor i does not change the probability by itself.

However, this does not mean complex phases are unimportant. Phases can affect later interference. Grover's algorithm depends heavily on this fact: it changes signs and phases of amplitudes so that later operations cause some amplitudes to grow and others to shrink.

4.4 Measurement in the computational basis

The basis

$$|0\rangle, |1\rangle$$

for one qubit, and more generally

$$|00 \cdots 0\rangle, |00 \cdots 1\rangle, \dots, |11 \cdots 1\rangle$$

for many qubits, is called the computational basis.

This is the standard basis used to encode classical bit strings in quantum circuits. For example, with three qubits, the computational basis states are

$$|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle.$$

A measurement in the computational basis asks:

> Which computational basis state do we observe?

For a general n -qubit state

$$|\psi\rangle = \sum_{x \in \{0,1\}^n} \alpha_x |x\rangle,$$

the probability of observing the bit string x is

$$P(x) = |\alpha_x|^2.$$

Here x ranges over all n -bit strings.

For example, a two-qubit state might be

$$|\psi\rangle = \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle.$$

Each amplitude has squared magnitude

$$\left|\frac{1}{2}\right|^2 = \frac{1}{4}.$$

So measurement gives each of the four outcomes with probability $1/4$:

$$P(00) = P(01) = P(10) = P(11) = \frac{1}{4}.$$

The negative sign in front of $|10\rangle$ does not change the immediate measurement probability in the computational basis. But it may affect what happens if we apply more quantum gates before measuring.

This is one of the central lessons of quantum computing:

> Amplitudes are not probabilities. Their signs and phases can matter before measurement, even when they do not change the final probability directly.

4.5 State collapse

Measurement does not only produce a classical outcome. It also changes the quantum state.

Suppose

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle.$$

If we measure in the computational basis and observe 0, then after measurement the state becomes

$$|0\rangle.$$

If we observe 1, then after measurement the state becomes

$$|1\rangle.$$

This update is called state collapse.

In the circuit model of quantum computation, collapse is the rule for updating the state after measurement (Nielsen and Chuang, 2010). It means that after a measurement outcome is obtained, the state becomes consistent with that outcome.

For example, if

$$|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}},$$

then measuring gives:

- outcome 0 with probability 1/2, after which the state is $|0\rangle$;
- outcome 1 with probability 1/2, after which the state is $|1\rangle$.

If we immediately measure again in the same basis, we get the same result with probability 1.

So if the first measurement gives 0, then the second measurement gives 0 for sure. The original superposition is no longer available.

This is very different from simply “looking up” a hidden classical value. Before measurement, the state was not just an unknown classical bit. It was a quantum superposition. Measurement turns it into a classical outcome and changes the state.

4.6 Collapse for multi-qubit states

For many qubits, the same principle applies.

Suppose we have the two-qubit state

$$|\psi\rangle = \frac{1}{\sqrt{3}}|00\rangle + \frac{1}{\sqrt{3}}|01\rangle + \frac{1}{\sqrt{3}}|11\rangle.$$

There is no $|10\rangle$ term, so the amplitude of $|10\rangle$ is 0.

The measurement probabilities are

$$P(00) = \frac{1}{3}, \quad P(01) = \frac{1}{3}, \quad P(10) = 0, \quad P(11) = \frac{1}{3}.$$

If measurement gives 01, then the post-measurement state becomes

$$|01\rangle.$$

Again, after the measurement, the original superposition is gone.

In Grover's algorithm, this is why we wait until the end to measure. If we measure too early, we destroy the amplitude pattern that the algorithm is trying to build.

4.7 Measuring only part of a quantum state

Sometimes we measure only some of the qubits.

Consider the two-qubit state

$$|\psi\rangle = \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle.$$

This is the equal superposition over all two-bit strings.

Suppose we measure only the first qubit.

To find the probability of observing first qubit 0, we add the probabilities of all basis states whose first bit is 0:

$$P(\text{first qubit} = 0) = P(00) + P(01).$$

Each of $|00\rangle$ and $|01\rangle$ has probability $1/4$, so

$$P(\text{first qubit} = 0) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Similarly,

$$P(\text{first qubit} = 1) = P(10) + P(11) = \frac{1}{2}.$$

If the measured first qubit is 0, then the state collapses to the part of the superposition consistent with first bit 0, renormalized:

$$\frac{|00\rangle + |01\rangle}{\sqrt{2}}.$$

If the measured first qubit is 1, then the state collapses to

$$\frac{|10\rangle + |11\rangle}{\sqrt{2}}.$$

The word renormalized means that we rescale the remaining state so that the total probability is again 1.

This rule generalizes. When we measure part of a quantum system, we keep the terms consistent with the observed outcome, discard the others, and then normalize the remaining state.

4.8 Example: measurement and correlation

Measurement becomes especially interesting when qubits are correlated.

Consider the two-qubit state

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

This is one of the standard Bell states used in quantum information theory (Nielsen and Chuang, 2010).

If we measure both qubits in the computational basis, we get:

$$P(00) = \frac{1}{2}, \quad P(11) = \frac{1}{2},$$

and

$$P(01) = 0, \quad P(10) = 0.$$

So the two qubits always agree. The outcome is random, but the two bits are perfectly correlated.

Now suppose we measure only the first qubit.

The probability of obtaining 0 is $1/2$, and the probability of obtaining 1 is $1/2$.

If the first measurement gives 0, the state collapses to

$$|00\rangle.$$

If the first measurement gives 1, the state collapses to

$$|11\rangle.$$

So after measuring the first qubit, the second qubit is no longer uncertain if measured in the same basis. It will match the first.

This example shows that measurement of one part of a multi-qubit state can affect the mathematical description of the whole state. This is not because a useful message has been sent faster than light. Rather, it is because the state describes correlations between measurement outcomes. The no-faster-than-light interpretation of such correlations is a standard point in quantum information theory (Mermin, 2007).

For Grover's algorithm, we will often work with registers containing several qubits. Measurement of the register produces one bit string. If the amplitudes have been amplified correctly, that bit string is likely to be a marked solution.

4.9 Measurement basis

So far, we have focused on measurement in the computational basis. But quantum states can be measured in other bases too.

A measurement basis is the set of mutually orthogonal states that define the possible outcomes of a measurement.

For one qubit, the computational basis is

$$|0\rangle, |1\rangle.$$

Another important basis is the plus-minus basis:

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}},$$

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

The states $|+\rangle$ and $|-\rangle$ are orthogonal because

$$\langle + | - \rangle = 0.$$

A measurement in the plus-minus basis asks:

> Is the state $|+\rangle$ or $|-\rangle$?

If the state is already

$$|+\rangle,$$

then measuring in the plus-minus basis gives outcome $+$ with probability 1.

If the state is

$$|-\rangle,$$

then measuring in the plus-minus basis gives outcome - with probability 1.

But what if the state is $|0\rangle$?

We can rewrite $|0\rangle$ in the plus-minus basis:

$$|0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}.$$

Therefore, measuring $|0\rangle$ in the plus-minus basis gives

$$P(+)=\frac{1}{2}, \quad P(-)=\frac{1}{2}.$$

This teaches an important point:

> Probabilities depend not only on the state, but also on the measurement basis.

In quantum circuits, measurements are usually drawn as computational-basis measurements. To measure in another basis, we typically apply a gate that changes basis, then measure in the computational basis. This is standard in the circuit model of quantum computation (Nielsen and Chuang, 2010).

For example, the Hadamard gate H satisfies

$$H|+\rangle = |0\rangle, \quad H|-\rangle = |1\rangle.$$

So measuring in the plus-minus basis can be done by applying H , then measuring in the computational basis.

4.10 Same probabilities, different states

Two quantum states can produce the same probabilities in one measurement basis but behave differently later.

Consider

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

and

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

If we measure either state in the computational basis, we get

$$P(0) = \frac{1}{2}, \quad P(1) = \frac{1}{2}.$$

So computational-basis measurement cannot distinguish them in a single direct measurement.

But the two states are not the same.

Apply a Hadamard gate:

$$H|+\rangle = |0\rangle,$$

while

$$H|-\rangle = |1\rangle.$$

Now measuring in the computational basis gives different deterministic outcomes.

This is the simplest example of why signs and phases matter. The states had the same immediate measurement probabilities, but their relative phases caused different interference after a later gate.

The phrase relative phase refers to phase differences between amplitudes in the same state. In

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}},$$

the two amplitudes have the same phase. In

$$\frac{|0\rangle - |1\rangle}{\sqrt{2}},$$

the amplitude of $|1\rangle$ differs by a phase of π , because multiplying by -1 is the same as multiplying by $e^{i\pi}$.

Grover's algorithm uses this principle directly. The oracle flips the phase of marked states. At first, a phase flip may seem invisible because it does not immediately change measurement probabilities. But after the diffusion operation, that phase difference becomes amplitude amplification.

4.11 Global phase does not matter

Relative phase matters. But global phase does not.

A global phase is a complex phase multiplying the entire state.

For example, consider

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle.$$

Now multiply the whole state by $e^{i\theta}$:

$$|\psi'\rangle = e^{i\theta}\alpha|0\rangle + e^{i\theta}\beta|1\rangle.$$

The probabilities in the computational basis are

$$P'(0) = |e^{i\theta}\alpha|^2 = |\alpha|^2,$$

and

$$P'(1) = |e^{i\theta}\beta|^2 = |\beta|^2.$$

More generally, global phase has no observable effect in quantum mechanics; physical states are represented up to an overall nonzero phase factor, usually taken as a unit complex number for normalized states (Nielsen and Chuang, 2010).

For example,

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

and

$$-\frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

represent the same physical state.

But

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

and

$$\frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

do not represent the same physical state, because the minus sign changes the relative phase between the two basis components.

4.12 Measurement is random, but probabilities are predictable

Quantum measurement is probabilistic. If

$$|\psi\rangle = \sqrt{0.9}|0\rangle + \sqrt{0.1}|1\rangle,$$

then one measurement does not guarantee outcome 0. It gives

$$P(0) = 0.9, \quad P(1) = 0.1.$$

A 10% event can still happen.

However, the probabilities are predictable. If we prepare the same state many times and measure each copy, the relative frequencies tend to match the Born-rule probabilities.

For example, if we prepare

$$|\psi\rangle = \sqrt{0.9}|0\rangle + \sqrt{0.1}|1\rangle$$

one thousand times, we expect roughly 900 outcomes of 0 and 100 outcomes of 1. The exact numbers will vary randomly.

In quantum computing experiments and simulators, repeated runs of the same circuit are often called shots.

A shot is one execution of a quantum circuit followed by measurement.

If a circuit is run S times, and an outcome appears k times, then the observed frequency is

$$\frac{k}{S}.$$

This frequency is an estimate of the true probability.

For example, suppose a one-qubit circuit is run 1000 times and produces:

- 0: 486 times,
- 1: 514 times.

Then the estimated probabilities are

$$\hat{P}(0) = 0.486, \quad \hat{P}(1) = 0.514.$$

This is consistent with a state that gives probabilities near $1/2$ and $1/2$.

Later, when we simulate Grover's algorithm, we will look at histograms of measured bit strings. A successful Grover circuit should produce marked solutions much more frequently than unmarked candidates.

4.13 Measurement does not reveal the whole quantum state

An n-qubit pure state can be written as

$$|\psi\rangle = \sum_{x \in \{0,1\}^n} \alpha_x |x\rangle.$$

There are 2^n amplitudes in this expression. It is tempting to think that a quantum computer stores 2^n pieces of directly readable information.

That is not correct.

When we measure n qubits in the computational basis, we get one n-bit string, not the full list of amplitudes.

For example, a three-qubit state has amplitudes for

$$|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle.$$

But one measurement gives only one result, such as

$$101.$$

It does not tell us the amplitudes of all eight basis states.

Also, after measurement, the original state has usually collapsed. We cannot simply measure again to get another independent sample from the same original state. To estimate the distribution, we need to prepare the state again and measure a fresh copy.

This is why quantum algorithms must be designed carefully. They cannot rely on reading out all amplitudes. Instead, they use interference to make useful answers likely before measurement.

Grover's algorithm is a perfect example:

1. Start with amplitudes spread evenly over all candidates.
2. Use an oracle to mark correct answers by changing their phase.
3. Use interference to increase the amplitudes of marked answers.
4. Measure only at the end.

The measurement still gives just one candidate, but the algorithm has shaped the probability distribution so that the candidate is likely to be correct.

4.14 Why not measure during the algorithm?

A natural beginner question is:

> If a quantum state contains amplitudes for many candidates, why not measure during the computation to learn something early?

The problem is that measurement can destroy the superposition and the phase relationships that the algorithm needs.

Imagine a search space with four candidates:

$$|00\rangle, |01\rangle, |10\rangle, |11\rangle.$$

A natural starting state for Grover's algorithm is the uniform superposition

$$|\psi\rangle = \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle.$$

If we measure immediately, each candidate appears with probability 1/4. We have gained no advantage over random guessing.

Grover's algorithm needs time to transform this state. It changes phases and then reflects amplitudes so that the marked candidate's amplitude grows.

If we measure too early, the state collapses to one candidate. The algorithm loses the amplitude pattern it was constructing.

This is one of the most important differences between classical and quantum computation:

> In a quantum algorithm, information can be useful before it is directly readable.

The amplitudes guide interference. They are not all individually accessible as output.

4.15 Measurement and Grover's algorithm

Let us connect this chapter directly to Grover search.

Suppose we have N candidates, represented by computational basis states

$$|0\rangle, |1\rangle, \dots, |N-1\rangle.$$

For simplicity, assume $N=2^n$, so the candidates fit exactly into n qubits.

Grover's algorithm prepares a state of the form

$$|\psi\rangle = \sum_{x=0}^{N-1} \alpha_x |x\rangle.$$

At the end, measuring gives candidate x with probability

$$P(x) = |\alpha_x|^2.$$

The goal is not to make the computer "look at every item" in a classical sense. The goal is to transform the amplitudes so that if w is a marked solution, then

$$|\alpha_w|^2$$

is large.

If there is one marked item, an ideal Grover algorithm can make the success probability close to 1 using about

$$O(\sqrt{N})$$

oracle calls, rather than $O(N)$ classical checks. We will derive this later. For now, the important measurement lesson is:

> Grover's algorithm wins by making the right answer likely to be measured.

The final measurement is simple. The hard part is everything before it.

4.16 A careful summary of the measurement rules

For the parts of quantum computing needed in this book, the following rules are enough.

A single-qubit state has the form

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,$$

where

$$|\alpha|^2 + |\beta|^2 = 1.$$

Measuring in the computational basis gives

$$0 \text{ with probability } |\alpha|^2,$$

and

$$1 \text{ with probability } |\beta|^2.$$

After measurement, the state becomes the observed basis state.

For an n-qubit state

$$|\psi\rangle = \sum_{x \in \{0,1\}^n} \alpha_x |x\rangle,$$

measuring all qubits in the computational basis gives outcome x with probability

$$|\alpha_x|^2.$$

After outcome x , the state becomes

$$|x\rangle.$$

If only part of the system is measured, probabilities are found by summing squared magnitudes over basis states consistent with the observed partial outcome. The state then collapses to the normalized part consistent with that outcome.

Measurement outcomes are random, but their probabilities are determined by the quantum state and the measurement basis.

Most importantly:

> Measurement gives limited classical information. Quantum algorithms must arrange amplitudes so that the limited information we can read is likely to be useful.

That sentence is the key to understanding Grover's algorithm.

4.17 Exercises

Exercise 1: Basic measurement probabilities

Let

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle.$$

Find

$$P(0)$$

and

$$P(1).$$

Exercise 2: Complex amplitudes

Let

$$|\psi\rangle = \frac{1}{\sqrt{5}}|0\rangle + \frac{2i}{\sqrt{5}}|1\rangle.$$

1. Check that the state is normalized.
2. Find the probability of measuring 0.
3. Find the probability of measuring 1.

Exercise 3: Two-qubit measurement

Let

$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle.$$

1. Find $P(00)$, $P(01)$, $P(10)$, and $P(11)$.
2. What is the probability that the first qubit is 1?
3. If the first qubit is measured and the result is 1, what is the new normalized state?

Exercise 4: Same computational probabilities

Consider

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

and

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

1. Show that both states give $P(0)=P(1)=1/2$ when measured in the computational basis.
2. Explain why the two states are nevertheless different quantum states.
3. What happens if a Hadamard gate is applied before measurement?

Exercise 5: Why Grover waits to measure

Suppose a Grover search begins with the uniform superposition over eight candidates:

$$|\psi\rangle = \frac{1}{\sqrt{8}} \sum_{x=0}^7 |x\rangle.$$

1. What is the probability of measuring any particular candidate immediately?
2. Why would measuring immediately fail to provide a search advantage?
3. In one sentence, explain why Grover's algorithm delays measurement until the end.

4.18 Chapter takeaway

Measurement is the point where quantum computation becomes classical output.

Before measurement, a quantum state is described by amplitudes. These amplitudes may be complex, negative, and capable of interference. During measurement, amplitudes produce probabilities through squared magnitudes. After measurement, the state collapses to match the observed outcome.

This creates both the power and the difficulty of quantum algorithms.

The power comes from manipulating amplitudes before they are observed.

The difficulty comes from the fact that measurement reveals only limited information and can destroy the superposition.

Grover's algorithm is built around this reality. It does not try to read all amplitudes. It uses phase changes and interference to make correct answers appear with high probability when measurement finally occurs.

In the next chapter, we will study quantum gates and circuits: the operations that change quantum states before measurement.

References

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Nielsen, Michael A., and Chuang, Isaac L. (2010). Quantum Computation and Quantum Information: 10th Anniversary Edition. Cambridge University Press.

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